

NON-REFLECTION OF THE BAD SET FOR  $\check{I}_\theta[\lambda]$  AND pcf

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ABSTRACT. We reconsider here the following related pcf questions and make some advances:

(Q1) concerning the ideal  $\check{I}_\kappa[\lambda]$  how much reflection do we have for the bad set  $S_{\lambda,\kappa}^{\text{bd}} \subseteq \{\delta < \lambda : \text{cf}(\delta) = \kappa\}$  assuming it is well defined, (for transparency only)?

(Q2) are there somewhat free black boxes?

The advances in (Q2) will be used in subsequent for constructions of Abelian groups and modules.

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## § 0. INTRODUCTION

§ 0(A). **Background.**

On  $\check{I}_\theta[\lambda]$  for  $\lambda > \theta$  regular see (Definition 0.12(3) and) [Sh:108], [Sh:88a], [Sh:420]. So we know that in many cases there is set  $S_{\lambda,\theta}^{\text{bd}} \subseteq S_\theta^\lambda := \{\delta < \lambda : \text{cf}(\delta) = \theta\}$  such that  $\text{dual}(\check{I}_\theta[\lambda]) = D_\lambda + (S_\theta^\lambda \setminus S_{\lambda,\theta}^{\text{bd}})$  and so  $S_{\lambda,\theta}^{\text{bd}}$  is unique (0.12(4)) modulo the club filter,  $\mathcal{D}_\lambda$ ; for definitions see §(0C).

We know that consistently, starting with a supercompact we can force that; e.g. GCH and  $S_{\aleph_{\omega+1}, \aleph_n}^{\text{bd}}$  (0.12(4)) is stationary for  $n = 1$  but we do not know it for  $n > 1$ . Still this set reflects in no  $\aleph_n$ , however we use G.C.H. or just  $\aleph_n > 2^{\aleph_0}$ . More generally, if  $\mu$  is strong limit of cofinality  $\aleph_0$  and  $S = S_{\mu^+, \aleph_1}^{\text{bd}}$  we do not know if  $S$  can reflect in stationarily many  $\delta$ 's of cofinality  $\aleph_n > \aleph_1$  when  $\aleph_n \leq 2^{\aleph_0}$ . Similarly for  $\mu$  strong limit of cofinality  $\kappa < \mu$ , (see 0.1, 0.2).

By [Sh:420, §1] for regular  $\lambda, \kappa$  such that  $\lambda > \kappa^+$  there is  $S \in \check{I}_\kappa[\lambda]$  which is stationary, in fact reflect in stationarily many  $\delta < \lambda$  of cofinality, e.g.  $\kappa^{+n} < \lambda$  for  $n \geq 1$  (check). Related subsets are the good/bad/chaotic sets of scales  $(\langle f_\alpha : \alpha < \lambda \rangle, f_\alpha \in {}^\kappa \mu)$ , see [Sh:g, Ch.II], [MgSh:204], [Sh:898] and 0.18 here.

The proof in [Sh:g, Ch.IX, §2] of  $\text{pp}(\aleph_\omega) < \aleph_{\omega_4}$  in particular continue these ideas.

Recall that if  $\bar{f} = \langle f_\alpha : \alpha < \lambda \rangle$  is  $<_J$ -increasing,  $<_J$ -cofinal in  $\prod_{i < \kappa} \lambda_i, \lambda_i =$

$\text{cf}(\lambda_i) > \theta \geq \kappa^+$  then  $S_\theta^{\text{gd}}(\bar{f}) := \{\delta < \lambda : \text{cf}(\delta) = \theta \text{ and } \bar{f} \restriction \delta \text{ is flat (see 0.18)}\}$  has complement orthogonal to  $\check{I}_\theta[\lambda]$  modulo the non-stationary ideal, (i.e. have a non-stationary intersection with any  $A \in \check{I}_\theta[\lambda]$ ).

We made this work after learning Kojman-Milovich-Spadaro [?], which shows  $\bar{f}$  is stable in  $\kappa^{+4}$ ; we learn later that  $S_\theta^{\text{gd}}(\bar{f}) = S_\theta^\lambda \text{ mod } D_\lambda$  when  $\theta = \kappa^{+n}, n \geq 4$  was pointed out, following the above, by Sharon-Viale [?, footnote 5], using Abraham-Magidor [AM10, 2.12, 2.19].

We start by continuing [Sh:420, §1], [Sh:g, Ch.IX, §2], to re-examine some of those problems; see §(0B). More specifically, we shed some light on question (Q1) in 0.1, 0.2 proved in §(1A).

What about (Q2)? This was a central issue of [Sh:898] using one dimensional. The  $n$ -dimensional are from [Sh:883] and lately [Sh:F1200], which relies on the results here; see 0.6, 0.4, 0.7 and proved in §(1B).

Much earlier Solovay proved that above a compact cardinal, the singular cardinal hypothesis holds; it follows that the so called strong hypothesis ( $\mu > \text{cf}(\mu) \Rightarrow \text{pp}(\mu) = \mu^+$ ) holds; so pcf becomes trivial. Moreover, by [Sh:g, Ch.II] if  $\text{pp}_J(\mu) > \lambda = \text{cf}(\lambda) > \mu > \text{cf}(\mu) = \kappa$  ( where  $J \supseteq [\kappa]^{<\kappa}$  is an ideal on  $\kappa$ ) then there is a sequence  $\langle f_\alpha : \alpha < \lambda \rangle$  with  $f_\alpha \in {}^\kappa \mu$  which is  $<_J$ -increasing and is  $\mu^+$ -free even as a sequence, so  $\bar{f} \restriction \delta$  is flat when  $\kappa < \text{cf}(\delta) < \mu$ , (i.e. the good set of  $\bar{f}$ ,  $\text{gd}(\bar{f})$  is large.

But if  $\kappa = \text{cf}(\mu) < \mu$ , the consistency result on  $\check{I}_{\kappa^+}[\mu^+]$  from [Sh:108] can be strengthened; we know consistently there are strong reflection properties say if GCH, consistently the case of Chang conjection holds from  $(\aleph_{\omega+1}, \aleph_\omega) \rightarrow (\aleph_1, \aleph_1)$ , by Levinski-Magidor-Shelah [LMSh:198] and  $(\aleph_{\omega+\omega+1}, \aleph_{\omega+\omega}) \rightarrow (\aleph_{\omega+1}, \aleph_\omega)$ . We can manipulate  $2^\kappa$  for  $\kappa$  regular.

§ 0(B). **Results.**

What do we accomplish? E.g. assume  $\lambda > \kappa > \aleph_0$  and for transparency assume  $S_{\lambda, \kappa}^{\text{bd}}$  is well defined. How much can it reflect? Assume  $\lambda = \mu^+$ ,  $\text{cf}(\mu) = \kappa$ ,  $\mu$  strong limit. We knew that ([Sh:108]) if, e.g.  $\theta = (2^\kappa)^{+n+1}$  then  $S_{\lambda, \kappa}^{\text{bd}}$  does not reflect in  $S_\theta^\lambda$ . Here 0.2 gives more: assuming  $(\forall n)(2^{\kappa^{+n}} < \lambda)$  we have, e.g. for  $n \geq 2, m \geq n+2$ : if  $S_{\lambda, \kappa}^{\text{bd}}$  reflect in  $S_{\kappa^{+n}}^\lambda$  this reflection does not reflect in  $S_{\kappa^{+m}}^\lambda$ ; moreover does not reflect in any  $S_{\theta^+}^\lambda, \theta \in \text{Reg} \cap \lambda \setminus \kappa^{+n+2}$ . See more in 0.2.

Returning to e.g. “if  $\bar{f}$  is  $<_J$ -increasing cofinal in  $\prod_{i < \kappa} \lambda_i/J$  and  $i < \kappa \Rightarrow \lambda_i = \text{cf}(\lambda_i) > \kappa$ ; how large is  $S_\theta^{\text{gd}}[\bar{f}]$ ”? We knew  $S_\theta^{\text{gd}}[\bar{f}]$  is large; here we prove in 0.1(1) that: if  $\theta \in [\kappa^{+4}, \kappa^{+\text{comp}(J)})$ ,  $(\forall i)(\theta < \lambda_i)$  and  $\theta$  is regular  $< \lambda$  then  $S_\theta^{\text{gd}}[\bar{f}]$  contains  $S_\theta^\lambda$  (modulo the club filter of course). Hence, e.g.  $\bar{f}$  is  $(\theta^{+\text{comp}(J)}, \theta^{+4}, J)$ -free when  $\kappa \leq \theta, \theta^{+\text{comp}(J)} < \min\{\lambda_i : i < \kappa\}$ , so if  $\lambda_\ell = \text{pp}(\mu_\ell) > \mu_\ell^+, \mu_\ell > \aleph_0 = \text{cf}(\mu_\ell)$  for  $\ell = 1, 2$  and  $\mu_1^{+4} \leq \lambda_1 < \lambda_2$  then  $(\lambda_2, \mu_2) \not\rightarrow (\lambda_1, \mu_1)$ . But this is not enough to prove what we need for Q2, i.e. 0.4 which is  $(\theta_2, \theta_1)$ -freeness; (the problem being for  $\langle \delta_i : i < \theta \rangle$  increasing continuous, for  $i$  of cofinality  $\leq \kappa$ ) but 1.11 tells us more, in particular, enough for Theorem 0.4.

More specifically, we shall show (the proofs are given later, the definitions appear in §(0C) below):

**Theorem 0.1.** *Assume  $\lambda > \sigma > \partial > \theta^+ > \theta > \aleph_0$  are regular.*

- 1) *Some  $S \in \tilde{I}_\theta[\lambda]$  reflect in every  $\delta \in S_\sigma^\lambda$ , see Definition 0.14(1).*
- 2) *Moreover, if  $\delta \in S_\sigma^\lambda$  then  $\{\delta_1 < \delta : \text{cf}(\delta_1) = \partial \text{ and } S \text{ reflects in } \delta_1\}$  is a stationary subset of  $\delta$ .*
- 3) *Moreover, for any  $(\partial, \theta, < \sigma)$ -system  $\bar{\mathcal{P}}^*$ , see Definition 0.9, for any ordinal  $\delta \in S_\sigma^\lambda$ , for any increasing continuous sequence  $\langle \delta_i : i < \sigma \rangle$  of ordinals with limit  $\delta$  (clearly exists) for some  $S_1 \in \tilde{I}_\partial^{\text{ac}}(\sigma, \sigma)$ , see Definition 0.13(2) we have:*

(\*) *if  $j \in S_\partial^\sigma \setminus S_1$  then there is  $S_2 \in I_\theta^{\text{cg}}(\bar{\mathcal{P}}^*)$  such that for some increasing continuous sequence  $\langle i_\varepsilon : \varepsilon < \partial \rangle$  with limit  $j$  we have  $\varepsilon \in S_\theta^\partial \setminus S \Rightarrow \delta_{i_\varepsilon} \in \text{good}_\theta''(\bar{\mathcal{P}})$ .*

**Theorem 0.2.** *Assume  $\lambda > \theta^{+\omega}$  and  $\lambda, \theta$  are regular uncountable and  $2^{\theta^{+n}} < \lambda$  for every  $n$ .*

- 1) *If  $S_{\lambda, \theta}^{\text{bd}}$  is (well defined and) stationary then there are  $n$  and stationary  $S \subseteq S_{\theta^{+n}}^\lambda$  which reflects in no ordinal  $\delta$  of cofinality  $\in [\theta, \theta^{+\omega})$ .*
- 2) *There is  $S \in \tilde{I}_\theta[\lambda]$  such that for every  $n \geq 2$ , either  $S_1 = S_{\theta^{+n}}^\lambda \cap \text{refl}(\lambda \setminus S)$  is not stationary (in  $\lambda$ ) or  $S_1$  is stationary but is the union of  $\leq 2^{\theta^{+n}}$  sets each of which reflect in no  $\delta$  of cofinality  $\in [\theta^{n+2}, \theta^{+\omega})$ .*
- 3) *In part (2) in the second possibility some stationary  $S_2 \subseteq S_1 (\subseteq S_{\theta^{+n}}^\lambda)$  either reflect in no ordinal of cofinality  $< \theta^{+\omega}$  or  $S_3 = \{\delta \in S_{\theta^{+n+1}}^\lambda : S_2 \cap \delta \text{ is stationary in } \delta\}$  is a stationary subset of  $S_{\theta^{+n+1}}^\lambda$  which reflect in no  $\delta < \lambda$  of cofinality  $< \theta^{+\omega}$ .*

In [Sh:898] we consider another version of freeness, note that being  $(\theta, \sigma)$ -free follows from  $\theta$ -free and is stronger than stable in every  $\kappa \in [\sigma, \theta)$ . We do not get it fully but enough to get “quite free **k**-combinatorial parameters” which is enough for applications in [Sh:F1200], [Sh:1006].

*Remark 0.3.* 1) Recall that for regular  $\partial > \aleph_0, \mu \in \mathbf{C}_\partial$  means  $\mu$  is strong limit singular of cofinality  $\partial$ .

- 2) For  $\partial = \aleph_0$  the above is almost equal to (and is contained in) the class  $\{\mu : \mu > \aleph_0 \text{ strong limit of cofinality } \aleph_0\}$ , more specifically, the difference does not reflect in any singular cardinal.
- 3) Having two possibilities in 0.4, make us prefer the non-tree version of the black box, (see [Sh:F1200]).

**Theorem 0.4.** Assume  $\sigma < \kappa$  are regular,  $\mu \in \mathbf{C}_\kappa$ , i.e.  $\mu$  is strong limit singular of cofinality  $\kappa$ .

At last one of the following holds:

- (A) there is a  $\mu^+$ -free  $\mathcal{F} \subseteq {}^\kappa\mu$  of cardinality  $\lambda := 2^\mu$ , this is called “ $\mu$  has a 1-solution”
- (B)  $\lambda = 2^\mu$  is regular and there is a  $(\lambda, \mu, \sigma \times \kappa) - 5$ -solution, see Definition 0.6.

**Claim 0.5.** If  $\mu > \kappa = \text{cf}(\mu) > \sigma = \text{cf}(\sigma)$  and we let  $\lambda = \mu^+$  then there is  $\bar{\eta}$  satisfying clauses (a)-(f) of Definition 0.6.

**Definition 0.6.** Assume  $\mu \in \mathbf{C}_\kappa$ ,  $\lambda = 2^\mu = \text{cf}(\lambda)$ ,  $\sigma = \text{cf}(\sigma) < \kappa$ ; we say  $\mathbf{x}$  is a  $(\lambda, \mu, \kappa, \sigma) - 5$ -solution when it consists of:

- (a)  $\bar{\eta} = \langle \bar{\eta}_\delta : \delta \in S \rangle$
- (b)  $S \subseteq S_\sigma^\lambda$  is stationary in  $\lambda$  (and  $\in \check{I}_\sigma[\lambda]$ )
- (c)  $\eta_\delta := \langle \alpha_{\delta,i,j} : (i,j) \in \sigma \times \kappa \rangle$  and  $\langle \alpha_{\delta,i,0} : i < \sigma \rangle$  is increasing with limit  $\delta$  and  $\alpha_{\delta,i,j} \in [\alpha_{\delta,i,0}, \alpha_{\delta,i,0} + \mu)$  increasing with  $j$  and  $\alpha_{\delta,i,0} + \mu \leq \alpha_{\delta,i+1,0}$ ; and let  $C_\delta = \{\alpha_{\delta,i,j} : (i,j) \in \sigma \times \kappa\}$
- (d) if  $\alpha_{\delta_1,i_1,j_1} = \alpha_{\delta_2,i_2,j_2}$  then  $(i_1, j_1) = (i_2, j_2)$  and  $i < i_1 \wedge j < j_2 \Rightarrow \alpha_{\delta_1,i,j} = \alpha_{\delta_2,i,j}$
- (e) [freeness]  $\bar{\eta}$  is  $(\theta^{+\kappa+1}, \theta^{+4}, J_*)$ -free, see 1.10(4) when  $\kappa \leq \theta < \mu$  and  $J_* = J_{\sigma \times \kappa}^{\text{bd}} = \{u \subseteq \sigma \times \kappa : \text{for some } (i_*, j_*) \in \sigma \times \kappa \text{ we have } u \subseteq \{(i,j) \in \sigma \times \kappa : i < i_* \text{ and } j < j_*\}\}$
- (f) [freeness]  $\bar{\eta}$  is  $(\kappa^+, J_*)$ -free
- (g) [black box] for every  $\chi < \mu$  and  $\bar{F} = \langle F_\delta : \delta \in S \rangle$  such that  $F_\delta : {}^{(C_\delta)}\delta \rightarrow \chi$  there is  $\bar{\alpha} = \langle \alpha_\delta : \delta \in S \rangle \in {}^S\chi$  such that  $(\forall \eta \in {}^\lambda\lambda)(\exists^{\text{stat}} \delta \in S)(\alpha_\delta = F(\eta \upharpoonright C_\delta))$ , e.g.
- (g)' for every relational vocabulary  $\tau$  of cardinality  $< \mu$  there is a sequence  $\bar{M} = \langle M_\delta \in S \rangle$ ,  $M_\delta$  a  $\tau$ -model with universe  $C_\delta := \text{Rang}(\eta_\delta) = \{\alpha_{\delta,i,j} : i < \sigma, j < \partial\}$  such that for every  $\tau$ -model  $M$  with universe  $\lambda$  we have  $(\exists^{\text{stat}} \delta \in S)(M_\delta = M \upharpoonright C_\delta)$ .

**Discussion 0.7.** 1) It may be helpful to use this to prove results by cases. First, find a proof using a 1-solution, that is with  $\mu^+$ -freeness using (A) of 0.4 or at least  $\theta_*$ -free,  $\mathcal{F} \subseteq {}^\kappa\mu$ ,  $|\mathcal{F}| = 2^\mu$ ,  $\theta_*$  large enough so in [Sh:F1200] terms using  $\mathbf{x}$  with  $\mathbf{k}_\mathbf{x} = 1$ . Second, use  $n$  cases of a 5-solution (see 0.4(B) and Definition 0.6) so have  $\mathbf{x} = \mathbf{x}_0 \times \mathbf{x}_1 \times \dots \times \mathbf{x}_n$ ,  $\mathbf{x}_\ell$  is as above so have enough cases of  $(\theta^\kappa, \theta^{+4})$ -freeness. This is done in [Sh:F1200] which uses Theorem 0.4.

2) We may use a different division to cases then 0.4, dividing case (B) as in [Sh:898]. Let  $\Upsilon = \min\{\partial : 2^\partial > 2^\mu\}$ ; and ask whether  $\Upsilon = \lambda$  or  $\Upsilon < \lambda$ .

2A) If  $\Upsilon = \lambda$  then  $\lambda = \lambda^{<\lambda}$  hence we have better statements on  $\lambda$ , e.g. if  $\lambda$  is a successor cardinal then we have  $\diamond_{S_{\aleph_0}^\lambda}$  or  $\diamond_{S_{\aleph_1}^\lambda}$  by [Sh:922].

2B) If  $\Upsilon < \lambda$ , by [Sh:898, §2], we can construct a black box for  $\Upsilon$  by [Sh:898, §2].

### § 0(C). Quoting Definitions.

We try to make this work reasonably self-contained.

*Notation* 0.8. 1) For regular uncountable cardinal  $\lambda$  let  $\mathcal{D}_\lambda$  be the filter generated by the clubs of  $\lambda$ .

2)  $\mathcal{H}(\chi)$  is the set of  $x$  with transitive closure of cardinality  $< \chi$ .

3) Let  $<_\chi^*$  will denote a well ordering of  $\mathcal{H}(\chi)$ .

4) For regular  $\kappa$  and cardinal (or ordinal)  $\lambda > \kappa$  let  $S_\kappa^\lambda = \{\delta < \lambda : \text{cf}(\delta) = \kappa\}$ .

5) For an ideal  $J$  on  $\kappa$  let  $\text{comp}(J)$  be  $\sup\{\theta : J \text{ is } \theta\text{-complete}\}$ .

**Definition 0.9.** 1) We say  $\bar{\mathcal{P}}$  is a  $(\partial, \theta, < \mu)$ -system when:

- (a)  $\theta \leq \partial$  are regular cardinals
- (b)  $\bar{\mathcal{P}} = \langle \mathcal{P}_\alpha : \alpha < \partial \rangle$
- (c) if  $a \in \mathcal{P}_\alpha$  then  $a \subseteq \alpha$  and  $|a| < \theta$
- (d)  $\beta \in a \in \mathcal{P}_\alpha \Rightarrow a_\alpha \cap \beta \in \mathcal{P}_\beta$
- (e)  $\mathcal{P}_\alpha$  has cardinality  $< \mu$ .

2) If  $\mu = \partial$  we may write  $(\partial, \theta)$ -system. Instead “ $< \mu^+$ ” we may write  $\mu$ . If  $\mathcal{P}_\alpha = \{a_\alpha\}$  for  $\alpha < \partial$  so  $\bar{\mathcal{P}}$  a  $(\partial, < \theta, 1)$ -system, and we may write  $\bar{a} = \langle a_\alpha : \alpha < \partial \rangle$  instead of  $\bar{\mathcal{P}}$ . Instead of  $\theta$  we may write  $\leq \partial$  when  $\theta = \partial^+$ .

*Remark* 0.10. Concerning Definition 0.9(1) note that we allow  $\mu > \partial$ ; in fact, this case was used in [Sh:g, Ch.II], in proving: if  $\lambda = \text{tcf}(\prod_{i < \kappa} \lambda_i, <_J)$ ,  $\lambda_i = \text{cf}(\lambda_i) > \kappa$  and  $\mu = \lim_J \langle \lambda_i : i < \kappa \rangle < \lambda_* = -\text{cf}(\lambda_*) < \lambda$  then there are  $\lambda_i^* = \text{cf}(\lambda_i^*) < \lambda_i$  with  $\mu = \lim_J \langle \lambda_i^* : i < \kappa \rangle$  such that  $\lambda_* = \text{tcf}(\prod_{i < \kappa} \lambda_i^*, <_J)$  exemplified by some  $\mu^+$ -free  $\langle f_\alpha : \alpha < \lambda_* \rangle$ .

*Fact* 0.11. For every regular  $\theta$  and stationary  $S \subseteq \{\delta < \theta^+ : \text{cf}(\delta) < \theta\}$  there is a  $(\theta^+, \theta, 1)$ -system, which means that there is  $\bar{a}$  satisfying:

- (a)  $\bar{a} = \langle a_\alpha : \alpha < \theta^+ \rangle$
- (b)  $a_\alpha \subseteq \alpha$
- (c)  $|a_\alpha| < \theta$
- (d)  $\beta \in a_\alpha \Rightarrow a_\beta = a_\alpha \cap \beta$
- (e) if  $E$  is a club of  $\theta^+$  and  $\zeta < \theta$  then there is  $\alpha$  such that  $a_\alpha \subseteq E \wedge \alpha = \sup(a_\alpha) \wedge \text{otp}(a_\alpha) = \zeta$
- (f) if  $E$  is a club of  $\theta^+$  and  $\zeta < \theta$ , then for some  $\delta \in S \cap E$  we have  $a_\delta \subseteq E \wedge \delta = \sup(a_\delta)$  and  $\zeta$  divides  $\text{otp}(a_\delta)$ .

*Proof.* See [Sh:g, Ch.III] + correction in [Sh:E12]. As of guessing clubs for clause (f), it is like [Sh:420, §1]. We just are more explicit in what we get.  $\square_{0.11}$

Recall ([Sh:108] = [Sh:88a], [Sh:420, §1]), (there we vary  $\theta$ )

**Definition 0.12.** 1) Let  $\lambda > \theta$  be regular.

2) For a  $(\lambda, \theta, < \mu)$ -system  $\bar{\mathcal{P}} = \langle \mathcal{P}_\alpha : \alpha < \lambda \rangle$  let

- $\text{good}'_{<\theta}(\bar{\mathcal{P}}) = \{\delta < \lambda : \text{cf}(\delta) < \theta \text{ and there is an unbounded } u \subseteq \delta \text{ of order type } < \delta \text{ such that } \alpha \in u \Rightarrow u \cap \alpha \in \mathcal{P}_\alpha\}$
- $\text{good}''_{<\theta}(\bar{\mathcal{P}})$  is defined similarly but  $\text{otp}(u) = \text{cf}(\delta)$ .

2A) For a  $(\lambda, \theta, < \mu)$ -system  $\bar{\mathcal{P}}$ , we define  $\text{good}'_{\leq\theta}(\bar{\mathcal{P}})$ ,  $\text{good}''_{\leq\theta}(\bar{\mathcal{P}})$  naturally; we defined  $\text{good}'_{=\theta}(\bar{\mathcal{P}})$ ,  $\text{good}''_{=\theta}(\bar{\mathcal{P}})$  similarly but demand  $\text{cf}(\delta) = \theta$  and add “ $u \in \mathcal{P}_\delta$ ”.

3)  $\check{I}_\theta[\lambda]$  is the set of  $S \subseteq S_\theta^\lambda := \{\delta < \lambda : \text{cf}(\delta) = \theta\}$  such that for some  $(\lambda, \theta, 1)$ -system  $\bar{a}$  and club  $E$  of  $\lambda$  we have  $S \cap E \subseteq \text{good}'_\theta(\bar{\mathcal{P}})$ , equivalently for some  $(\lambda, < \theta, 1)$ -system  $\bar{a}$  and club  $E$  of  $\lambda$ ,  $S \cap E \subseteq \text{good}''_\theta(\bar{a})$ ; equivalently, we may use  $\bar{\mathcal{P}}$  a  $(\lambda, \lambda, < \lambda)$ -system or  $(\lambda, \theta, < \lambda)$ -system; abusing notation for  $S \subseteq \lambda$ ,  $S \in \check{I}_\theta[\lambda]$  means  $S \cap S_\theta^\lambda \in \check{I}_\theta[\lambda]$ .

4) If  $\check{I}[\lambda] = (\text{the non-stationary ideal on } S_\theta^\lambda) + S_*$  then we call  $S_*$  the good set on  $\lambda$  for cofinality  $\theta$ ; it will be denoted  $S_{\lambda,\theta}^{\text{gd}}$ ; its complement  $S_{\lambda,\theta}^{\text{bd}} := S_\theta^\lambda \setminus S_*$  is called the bad set; of course, as only  $S_*/\mathcal{I}_\lambda$  is unique this notation pedantically is not justified.

5) Let  $\check{I}_\kappa^\perp[\lambda] = \{S \subseteq S_\kappa^\lambda : \text{if } S_1 \in \check{I}_\kappa[\lambda] \text{ then } S_1 \cap S \text{ is not stationary (in } \lambda)\}$ .

6) Let  $\check{I}[\lambda] = \{S \subseteq \lambda : \text{if } \theta = \text{cf}(\theta) < \lambda \text{ then } S \cap S_\theta^\lambda \in \check{I}_\theta[\lambda]\}$ .

**Definition 0.13.** Let  $\lambda > \theta^+$  be regular.

1) Let  $I_\theta^{\text{cg}}[\lambda, \mu]$  be the set of  $S \subseteq S_\theta^\lambda$  such that (cg stands for club guessing) there is no  $\bar{\mathcal{P}}$  witnessing  $S \in (I_\theta^{\text{cg}}[\lambda, \mu])^+$  which means  $S \subseteq S_\theta^\lambda \wedge S \notin I_\theta^{\text{cg}}(\bar{\mathcal{P}})$  that is:

- (\*)<sub>1</sub>  $\bar{\mathcal{P}} = \langle \mathcal{P}_\alpha : \alpha < \lambda \rangle$  is a  $(\lambda, \theta, < \mu)$ -system
- (\*)<sub>2</sub> for  $\bar{\mathcal{P}}, \lambda$  as above let  $I_\theta^{\text{cg}}(\bar{\mathcal{P}})$  be the set of  $S \subseteq \lambda$  such that
  - for some club  $E$  of  $\lambda$  for no  $\delta \in S$  and  $a \in \mathcal{P}_\delta$  do we have  $a \subseteq E \wedge \sup(a) = \delta$ .

1A) We define  $I_\theta^{\text{dg}}[\lambda, \mu], I_\theta^{\text{dg}}(\bar{\mathcal{P}})$  similarly except that in • of (\*)<sub>2</sub> we demand only  $a \in \mathcal{P}_{<\lambda}$ .

2) Assume  $\lambda = \text{cf}(\lambda) \geq \theta = \text{cf}(\theta), \lambda \geq \mu, \mu^+ \geq \theta$ . Let  $\check{I}_\theta^{\text{ac}}\langle \lambda, \mu \rangle$  be the set of  $S \subseteq S_\theta^\lambda$  such that there are  $\chi > \lambda + \mu$  and  $x \in \mathcal{H}(\chi)$  for which there is no sequence  $\bar{N} = \langle N_\varepsilon : \varepsilon < \theta \rangle$  satisfying:

- (a)  $N_\varepsilon \prec (\mathcal{H}(\chi), \theta, <_\chi^*)$
- (b)  $\langle N_\zeta : \zeta < \theta \rangle$  is increasing continuous
- (c)  $\langle N_\zeta : \zeta \leq \varepsilon \rangle \in N_{\varepsilon+1}$
- (d)  $\|N_\varepsilon\| < \mu$  and  $N_\varepsilon \cap \mu$  is an ordinal
- (e)  $\{x, \lambda, \mu, \theta\} \in N_0$
- (f)  $\cup\{N_\varepsilon \cap \lambda : \varepsilon < \theta\} \in S$ .

**Definition 0.14.** For  $\lambda$  regular uncountable and unbounded  $S \subseteq \lambda$  let  $\text{refl}(S) = \{\delta < \lambda : \text{cf}(\delta) > \aleph_0 \text{ and } S \text{ reflects in } \delta\}$  where “ $S$  reflects in  $\delta$ ” means  $S \cap \delta$  is a stationary subset of  $\delta$ .

2) We say  $S \subseteq \lambda$  reflects in  $S_\theta^\lambda$  if  $\{\delta \in S_\theta^\lambda : S \cap \delta \text{ is stationary in } \delta\}$  is a stationary subset of  $\lambda$ . We may replace  $S_\theta^\lambda$  by any stationary subset of  $\lambda$ .

**Definition 0.15.** For a regular cardinal  $\partial$ , let  $\mathbf{C}_\partial$  be the class of strong limit singular cardinals  $\mu$  of cofinality  $\partial$  such that  $\text{pp}^*(\mu) = {}^+ 2^\mu$ .

*Fact 0.16.* 1) Assume  $\lambda$  is regular and  $\lambda = \text{cf}(\lambda) > \mu$  and if  $\lambda = \mu^+$ ,  $\mu$  regular, or  $\alpha < \lambda \Rightarrow \text{cf}([\alpha]^{<\mu}, \subseteq) < \lambda$  and  $\mu^+ < \lambda$ , then  $\theta = \text{cf}(\theta) < \mu \Rightarrow S_\theta^\lambda \in \check{I}_\theta[\lambda]$ , moreover, there is a closed  $(\lambda, \mu, < \lambda)$ -system  $\bar{\mathcal{P}}$  such that:  $\delta < \lambda \wedge \text{cf}(\delta) < \mu \Rightarrow (\exists a \in \mathcal{P}_\delta)(\sup(a) = \delta \wedge \text{otp}(a) = \text{cf}(\delta))$ .

2)  $\check{I}_\theta^{\text{ac}}(\lambda, \mu) \subseteq \check{I}_\theta[\lambda]$  when defined.

3) If  $\lambda > \theta^+$  and  $\lambda, \theta$  are regular then there is a  $(\lambda, \leq \theta, < \lambda)$ -system  $\bar{\mathcal{P}}$  such that  $S_\theta^\lambda \notin I_\theta^{\text{cg}}(\bar{\mathcal{P}})$  and  $\text{otp}(a) = \theta$ .

**Discussion 0.17.** 1) For the equivalence of the two versions in Definition 0.12(3), see [Sh:420, §1].

2) When does  $S_{\lambda, \theta}^{\text{gd}}$  exist?

See [Sh:108] = [Sh:88a],  $S_{\lambda, \theta}^{\text{gd}}$  exists under quite weak cardinal arithmetic assumptions (much weaker than GCH).

3) Of course, if  $\alpha < \lambda \Rightarrow |\alpha|^{<\theta} < \lambda$  then  $S_{\lambda, \theta}^{\text{bd}} = \emptyset$ .

4) It is proved there for  $\lambda$ , e.g. successor of strong limit singular  $\mu$  and  $\theta \in (\text{cf}(\mu), \mu)$  that  $S_{\lambda, \theta}^{\text{bd}}$  exists and does not reflect in cofinality  $(2^\theta)^+$  and in cofinality  $\partial$  when  $(\forall \alpha < \partial)[|\alpha|^\theta < \partial]$ .

5) Also it is proved ([Sh:g, Ch.II]) that if  $\lambda$  is a successor of regular  $\aleph_0 < \theta = \text{cf}(\theta)$  and  $\theta^+ < \lambda$  then  $S_{\lambda, \theta}^{\text{bd}}$  is  $\emptyset$ ; (i.e. not stationary), see 0.11 or 0.16(1).

In §(1B) we shall use [Sh:g, Ch.II].

**Definition 0.18.** Let  $\bar{f}$  be  $<_J$ -increasing in  ${}^\kappa\text{Ord}$ ,  $J$  an ideal on  $I$ .

1) We say  $\bar{f}$  is flat in  $\delta$  or  $\delta \in S_{\text{gd}}[\bar{f}, J] = S_J^{\text{gd}}[\bar{f}]$  when  $\delta \leq \ell g(\bar{f})$ ,  $\text{cf}(\delta) > \kappa$  and there is a  $<_J$ -eub  $g$  to  $\bar{f} \restriction \delta$  such that  $(\forall i < \kappa)(\text{cf}(g(i)) = \text{cf}(\delta))$ , equivalently there are increasing sequences  $\langle \alpha_{i, \varepsilon} : \varepsilon < \text{cf}(\delta) \rangle$  for  $i < \kappa$  such that  $(\forall \alpha < \delta)(\exists \varepsilon < \text{cf}(\delta))(f_\alpha <_J \langle \alpha_{i, \varepsilon} : i < \kappa \rangle)$  and  $(\forall \varepsilon < \text{cf}(\delta))(\exists \alpha < \delta)(\langle \alpha_{i, \varepsilon} : i < \kappa \rangle <_J f_\alpha)$ .

2) We say  $\delta$  is strongly chaotic for  $\bar{f}$  or  $\delta \in S_{\text{sch}}[\bar{f}, J] = S_J^{\text{sch}}[\bar{f}]$  when there is a sequence  $\langle u_i : i < \kappa \rangle$ ,  $u_i \subseteq \text{Ord}$ ,  $|u_i| \leq \kappa$  and  $(\forall \alpha < \delta)(\exists g \in \prod_i u_i)(\exists \beta < \delta)(f_\alpha <_J g <_J f_\beta)$ .

2A) We say  $\delta$  is chaotic for  $\bar{f}$  or  $\delta \in S_J^{\text{ch}}[\bar{f}] = S_{\text{ch}}[\bar{f}, J]$  when there is  $\bar{u}$  as above such that for every  $\alpha < \delta$  for some  $\beta \in (\alpha, \delta)$  the set  $A_{\alpha, \beta} = A_{\alpha, \beta}[\bar{u}, \bar{f}]$  belongs to  $J^+$  where  $A_{\alpha, \beta} = \{i < \kappa : \min(u_i \cup \{\infty\} \setminus f_\alpha(i)) < \min(u_i \cup \{\infty\} \setminus f_\beta(i))\}$ .

2B) We define  $S_\theta^{\text{sch}}[\bar{f}, J] = S_{J, \theta}^{\text{sch}}[\bar{f}]$ ,  $S_\theta^{\text{ch}}[\bar{f}, J] = S_{J, \theta}^{\text{ch}}[\bar{f}]$  similarly but restricting ourselves to  $\delta$  of cofinality  $\theta$ .

3) We say  $\delta$  is bad for  $\bar{f}$  or  $\delta \in S_{\text{bd}}[\bar{f}, J] = S_J^{\text{bd}}[\bar{f}]$  when  $\delta \leq \ell g(\bar{f})$ ,  $\text{cf}(\delta) > \kappa$  and  $\bar{f} \restriction \delta$  has  $<_J$ -eub  $g$  but is not flat.

**Claim 0.19.** Let  $J, \bar{f}$  be as in 0.18.

1) If  $\delta \leq \ell g(\bar{f})$ ,  $\text{cf}(\delta) > \kappa^+$  then  $\delta$  satisfies exactly one of good, bad or chaotic.

2) In other words  $\{\delta : \delta \leq \ell g(\bar{f}) \text{ and } \text{cf}(\delta) > \kappa^+\}$  is included in the disjoint union of  $S_{\text{gd}}[\bar{f}], S_{\text{bd}}[\bar{f}], S_{\text{sch}}[\bar{f}]$ .

*Proof.* By [Sh:g, Ch.II, §2]. □??

**Claim 0.20.** Let  $\bar{f}, J, \kappa$  be as in 0.18 and  $\lambda = \ell g(\bar{f})$ .

1) If  $\delta \in S_J^{\text{ch}}[\bar{f}]$  then for some club  $e$  of  $\delta$ , we have  $\alpha \in e \wedge \text{cf}(\alpha) > \kappa \Rightarrow \alpha \in S_J^{\text{ch}}[\bar{f}]$ .

1A) Similarly for  $S_{\text{sch}}[\bar{f}]$ .

2) If  $\delta \in S_J^{\text{gd}}[\bar{f}]$  then for some club  $e$  of  $\delta$  we have  $\alpha \in e \wedge \text{cf}(\alpha) > \kappa \Rightarrow \alpha \in S_J^{\text{gd}}[\bar{f}]$ .

3) If  $\delta \leq \lambda$ ,  $\text{cf}(\delta) \in S_{\text{bd}}[\bar{f}]$  then  $\text{cf}(\delta) \geq \kappa^{+\text{comp}(J)+1}$ .



**Claim 0.21.** *Assume  $(\lambda, \bar{\lambda}, J, \kappa)$  is a pcf case,  $\bar{f}$  a witness for it, see Definition 1.6. If  $\kappa < \sigma < \min\{\lambda_i : i < \kappa\}$  or just  $\kappa < \sigma < \lim - \inf_J(\bar{\lambda})$  and  $S \in \check{I}_\sigma[\lambda]$  then  $E \cap S \subseteq S_{\text{gd}}[\bar{f}]$  for some club  $E$  of  $\lambda$ .*

## § 1. ON SYSTEMS

§ 1(A). Existence of large members of  $\check{I}_\theta[\lambda]$ .

**Claim 1.1.** Assume  $\lambda > \aleph_1$  is regular and  $M_* \prec (\mathcal{H}(\lambda), \in)$  has cardinality  $< \lambda$  and  $\{\lambda, \theta\} \subseteq M_*$  and  $M_* \cap \lambda \in \lambda$ . Then we can find a pair  $(E, \bar{\mathcal{P}})$  which is  $(\lambda, M_*)$ -suitable, which means:

- ⊞ (a)  $E$  is a club of  $\lambda$ ; we may add  $\alpha \in E \wedge \alpha > \sup(\alpha \cap E) \Rightarrow \text{cf}(\alpha) = \aleph_0$
- (b)  $\bar{\mathcal{P}} = \langle \mathcal{P}_\alpha : \alpha < \lambda \rangle$  is a  $(\lambda, \lambda, < \lambda)$ -system and  $\theta = \text{cf}(\theta) < \lambda \cap M_* \Rightarrow \text{good}_\theta''(\bar{\mathcal{P}}) \supseteq S_\theta^\lambda \setminus E$
- (c) if  $\sigma > \partial$  are regular  $\in \lambda \cap M_*$  and  $\bar{\mathcal{P}}^* = \langle \mathcal{P}_\alpha^* : \alpha < \partial \rangle \in M_*$  is a  $(\partial, \partial, < \sigma)$ -system and  $\langle \delta_i : i < \sigma \rangle$  is an increasing continuous sequence of members from  $E$ , then there are  $f, e$  such that:
  - (α)  $e$  is a club of  $\partial$
  - (β)  $f$  is an increasing continuous function from  $\partial$  into  $\{\delta_i : i < \sigma\}$
  - (γ) if  $\varepsilon < \partial, a \in \mathcal{P}_\varepsilon^*$  and  $a \subseteq e$  then  $\{f(\xi) : \xi \in a \text{ and } \text{otp}(a \cap \xi) \text{ is a successor ordinal}\} \in \mathcal{P}_{f(\varepsilon+1)}$
  - (c)<sup>+</sup> like (c) but we replace (γ) by
    - (γ)<sup>+</sup> if  $\varepsilon < \partial, a \in \mathcal{P}_\varepsilon^*$  and  $a \subseteq e$  and  $\langle \gamma_\iota : \iota < \text{otp}(a) \rangle$  list  $a$  in increasing order then in addition to the conclusion of (γ)
      - we can choose  $\beta_\iota \in [\gamma_\iota, \gamma_{\iota+1})$  for  $\iota < \text{otp}(a)$  such that  $\{\beta_j : j \leq \iota\} \in \mathcal{P}_{\beta_{\iota+1}}$  for every  $\iota < \text{otp}(a)$
      - if  $a$  has no last member then  $\sup(a) \in \text{good}_\theta''(\bar{\mathcal{P}})$
- (d) if  $\langle \delta_i : i < \sigma \rangle$  is an increasing continuous sequence of members of  $E$  and  $\sigma > \partial > \theta$  are regular  $\in \lambda \cap M_*$  and  $\bar{\mathcal{P}}^* = \langle \mathcal{P}_\varepsilon^* : \varepsilon < \partial \rangle \in M_*$  is a  $(\partial, \leq \theta, < \sigma)$ -system then for some  $e, f$  satisfying clauses (α), (β), (γ), (γ)<sup>+</sup> we have
- (δ) the following set belongs to  $I_\theta^{\text{dg}}(\bar{\mathcal{P}}^*)$ , recalling 0.13(1A)
 
$$\{\zeta \in S_\theta^\partial : \text{there is no } a \subseteq e, a \in \mathcal{P}_{<\partial}^* \text{ such that } a \subseteq \zeta = \sup(a) \text{ and } \text{otp}(a) = \theta\}$$
- (ε) the following set belongs to  $\check{I}_\theta^{\text{ac}}\langle \sigma, \sigma \rangle$ , see Definition 0.13(2)
 
$$\{i \in S_\theta^\sigma : \text{there are no } e, f \text{ satisfying } \sup(e) = i \text{ and clauses } (\alpha), (\beta), (\gamma), (\gamma)^+, (\delta) \text{ above}\}.$$

*Remark 1.2.* 1) Note that for  $\text{good}_\theta''(\bar{\mathcal{P}})$ , only  $\langle \mathcal{P}_\alpha \cap [\alpha]^{<\theta} : \alpha < \lambda \rangle$  matters.  
 2) For  $\bar{M}$  as in  $\odot_1$  in the proof and  $\alpha < \lambda$  essentially  $\bar{\mathcal{P}}$  satisfies the conclusion with  $M_*$  replaced by  $M_\alpha$ , the essentially because we should ignore the ordinals  $\leq \alpha$ , i.e. in clauses (c), (c)<sup>+</sup>, (d) demand  $\delta_0 > \alpha$ .

*Proof.* Let  $\chi > \lambda$  and let  $\bar{M}$  be such that:

- $\odot_1$  (a)  $\bar{M} = \langle M_\alpha : \alpha < \lambda \rangle$  be a  $\prec$ -increasing continuous sequence
- (b)  $M_\alpha \prec (\mathcal{H}(\chi), \in, <_\chi^*)$
- (c)  $\|M_\alpha\| < \lambda$
- (d)  $\bar{M} \restriction (\alpha + 1) \in M_\alpha$

- (e)  $M_\alpha \cap \lambda \in \lambda$  for every  $\alpha < \lambda$
- (f)  $\bar{\mathcal{P}}^* \in M_0$
- (g) moreover  $M_* \in M_0$  hence  $M_* \subseteq M_0$ .

Let  $E = \{\alpha : M_\alpha \cap \lambda = \alpha\}$ . Clearly  $E$  is a club of  $\lambda$ , hence clause (a) of  $\boxplus$  holds, even the “we may add”.

Let  $\bar{\mathcal{P}} = \langle \mathcal{P}_\alpha : \alpha < \lambda \rangle$  be defined by:

- $\odot_2$   $\mathcal{P}_\alpha = \{a \in M_{\alpha+1} : a \subseteq \alpha \text{ so } |a| < \lambda \text{ and } \beta \in a \Rightarrow a \cap \beta \in M_{\beta+1}\}$  so  $\bar{\mathcal{P}} = \langle \mathcal{P}_\alpha : \alpha < \lambda \rangle$  is a  $(\lambda, \lambda, < \lambda)$ -system, moreover,  $\boxplus(b)$  holds.

[Why does  $\boxplus(b)$  hold? Let  $\delta \in S_\theta^\lambda \setminus E$  be a limit ordinal, so for some  $\alpha < \delta$  we have  $\delta \in M_\alpha$  hence there is an unbounded (and even closed) subset  $a$  of  $\delta$  in  $M_\alpha$  of order type  $\text{cf}(\delta)$  so  $\beta \in (a \setminus \alpha) \Rightarrow (a \setminus \alpha) \cap \beta \in M_\alpha \subseteq M_\beta \Rightarrow (a \setminus \alpha) \cap \beta \in M_\beta$ . So indeed  $\text{good}_\theta''(\bar{\mathcal{P}}) \supseteq S_\theta^\lambda \setminus E$ .]

So we arrive to the main point, that is to prove clauses (c),  $(c)^+$  and later comment on its relative (d). So let  $\partial < \sigma \in M_* \cap \lambda$  be regular and  $\bar{\mathcal{P}}^* \in M_*$  be a  $(\partial, \partial, < \sigma)$ -system and let  $\bar{\delta} = \langle \delta_i : i < \sigma \rangle$  be an increasing continuous sequence of ordinals from  $E$  and let  $\delta_\sigma := \cup\{\delta_i : i < \sigma\}$  so also  $\langle \delta_i : i \leq \sigma \rangle$  is an increasing continuous sequence of ordinals from  $E$ .

We choose  $N_\varepsilon$  by induction on  $\varepsilon \leq \partial$  such that:

- $\odot_3$  (a)  $N_\varepsilon \prec (\mathcal{H}(\chi), \in, <_\chi^*)$
- (b)  $\|N_\varepsilon\| < \sigma$
- (c)  $\langle N_\xi : \xi \leq \zeta \rangle \in N_\varepsilon$  when  $\zeta < \varepsilon$
- (d)  $\langle N_\zeta : \zeta \leq \varepsilon \rangle$  is  $\prec$ -increasing continuous
- (e)  $\lambda, \sigma, \partial, \theta, E, \bar{M}, \bar{\delta}$  and  $\bar{\mathcal{P}}^*$  belongs to  $N_\varepsilon$
- (f)  $\partial + 1 \subseteq N_\varepsilon$  moreover (follows if  $\sigma = \partial^+$ )  $N_\varepsilon \cap \sigma \in (\partial, \sigma)$ .

This is easy. Let  $i(\varepsilon) := N_\varepsilon \cap \sigma$  for  $\varepsilon \leq \partial$ , hence  $i(\varepsilon) < \sigma$  is increasing continuous with  $\varepsilon$ . So  $\delta_{i(\varepsilon)}$  is an ordinal  $\in E \subseteq \lambda$  hence  $M_{\delta_{i(\varepsilon)}}$  is well defined and  $\delta_{i(\varepsilon)} \in M_{\delta_{i(\varepsilon)}+1}$ , also  $\langle \delta_{i(\varepsilon)} : \varepsilon < \partial \rangle$  is increasing continuous with limit  $\delta_{i(\partial)}$ . For  $\varepsilon = \partial$  clearly  $\text{cf}(\delta_{i(\varepsilon)}) = \text{cf}(\delta_{i(\partial)}) = \text{cf}(\partial) = \partial$  hence

- $\oplus_1$  (a) there is a club  $C$  of  $\delta_{i(\partial)}$  of order type  $\text{cf}(\delta_{i(\partial)}) = \partial$
- (b) necessarily  $C \in \mathcal{H}(\chi)$  and without loss of generality  $C \in M_{\delta_{i(\partial)}+1}$
- (c) let  $g$  be the unique increasing continuous function from  $\partial$  onto  $C$ , so necessarily  $g \in M_{\delta_{i(\partial)}+1}$
- (d) let  $e = \{\varepsilon < \partial : \delta_{i(\varepsilon)} \in C, \text{ moreover } \varepsilon = \text{otp}(C \cap \delta_{i(\varepsilon)}) \text{ and, actually follows, } \delta_{i(\varepsilon)} = g(\varepsilon)\}$
- (e) let  $f : \partial \rightarrow \sigma$  be defined by  $f(\varepsilon) = \delta_{i(\varepsilon)}$ .

Now  $C$  is a club of  $\partial$  and both  $\langle g(\varepsilon) : \varepsilon < \partial \rangle$  and  $\langle \delta_{i(\varepsilon)} : \varepsilon < \partial \rangle$  are increasing continuous sequences of ordinals with limit  $\delta_{i(\partial)}$ , so clearly

- $\oplus_2$   $e$  is a club of  $\partial$ .

So concerning clause (c) (of  $\boxplus$ ) it suffices to prove that the pair  $(f, e)$  we have just chosen is as required there. Now obviously  $e, f$  satisfy sub-clauses  $(\alpha), (\beta)$  of (c). What about sub-clause  $(\gamma)$  of clause (c) and subclause  $(\gamma)^+$  of clause (c)<sup>+</sup>?

Clearly

$$\oplus_3 \quad f \restriction e = g \restriction e, \text{ see the definition of } e.$$

Now we shall prove

$$\oplus_4 \quad \text{if } \varepsilon < \partial \text{ and } a \in \mathcal{P}_\varepsilon^* \text{ satisfies } a \subseteq e, \text{ then } \{g(\zeta) : \zeta \in a\} \in M_{f(\varepsilon+1)}.$$

The proof of  $\oplus_4$  is done in  $(*)_{4.1} - (*)_{4.7}$ .

Note

$$(*)_{4.1} \quad \mathcal{P}_\varepsilon^* \subseteq N_0 \cap M_0 \subseteq N_{\varepsilon+1} \cap M_{\delta(\partial)+1} \subseteq N_{\varepsilon+1} \cap M_{\delta_\sigma}.$$

[Why? Obvious as  $\bar{\mathcal{P}}^* \in M_*$ ,  $\partial = \ell g(\bar{\mathcal{P}}^*) \in M_* \cap \lambda$  but  $M_* \cap \lambda \subseteq M_0 \cap \lambda \in \lambda$  hence  $\partial \subseteq M_0$  so together  $\mathcal{P}_\varepsilon^* \in M_0$ . Now  $|\mathcal{P}_\varepsilon^*| < \sigma < \lambda$  and  $\sigma \in M_* \cap \lambda \subseteq M_0 \cap \lambda \in \lambda$  so  $\mathcal{P}_\varepsilon^* \subseteq M_0 \subseteq M_{\delta_{i(\varepsilon)}} \subseteq M_{i(\partial)} \subseteq M_{\delta_\sigma}$ . Also  $\bar{\mathcal{P}}^* \in N_0$  and  $\varepsilon, \partial \in N_{\varepsilon+1}$  and  $|\mathcal{P}_\varepsilon^*| + \partial < \sigma$  and by  $\odot_3(f)$  we have  $N_{\varepsilon+1} \cap \sigma \in \sigma$  hence  $\mathcal{P}_\varepsilon^* \subseteq N_{\varepsilon+1}$ , so together we are done.]

Also

$$(*)_{4.2} \quad \{g(\zeta) : \zeta \in a\} \in M_{\delta_{i(\partial)+1}} \prec M_{\delta_\sigma}.$$

[Why? As  $a$  and  $g$  belong to this model; why? For  $a$  because  $a \in \mathcal{P}_\varepsilon^*$ , see the assumption of  $\oplus_4$  and  $\mathcal{P}_\varepsilon^* \subseteq M_0 \subseteq M_{\delta_{i(\partial)}} \subseteq M_{\delta_{i(\partial)+1}}$  by  $(*)_{4.1}$ . For  $g$ , by the choice of  $C$  and  $g$ , see  $\oplus_1(a), (b), (c)$ .]

$$(*)_{4.3} \quad \{g(\zeta) : \zeta \in a\} = \{(f \restriction \varepsilon)(\zeta) : \zeta \in a\} \in N_{\varepsilon+1}.$$

[Why? The equality holds by  $\oplus_3$  as  $a \subseteq e \wedge a \subseteq \varepsilon$  by the assumption of  $\oplus_4$ . The membership “ $\in N_{\varepsilon+1}$ ” holds as on the one hand  $a \subseteq \varepsilon, a \in \mathcal{P}_\varepsilon^*$  hence by  $(*)_{4.1}$  also  $a \in N_{\varepsilon+1}$  and on the other hand  $f \restriction \varepsilon \in N_{\varepsilon+1} \prec N_\partial$  as  $\langle N_\zeta : \zeta \leq \varepsilon \rangle \in N_{\varepsilon+1}$  by  $\odot_3(c)$  hence  $i(\zeta) : \zeta \leq \varepsilon \in N_{\varepsilon+1}$  by the choice  $i(\zeta) = \sup(N_\zeta \cap \sigma)$  after  $\odot_3$  and  $\bar{\delta} \in N_0$  by  $\odot_3(e)$  hence  $\langle \delta_{i(\zeta)} : \zeta \leq \varepsilon \rangle \in N_{\varepsilon+1}$  so  $f \restriction (\varepsilon + 1) \in N_{\varepsilon+1}$  by  $\oplus_2(e)$ .]

As  $\bar{\delta} \in N_0 \prec N_{i(\partial)}$  by  $\odot_3(e)$  we have  $\bar{\delta} = \langle \delta_i : i \leq \sigma \rangle \in N_0 \prec N_{\varepsilon+1}$  so necessarily  $\delta_\sigma \in N_0 \prec N_{\varepsilon+1}$  and recalling  $\bar{M} \in N_0$  by  $\odot_3(e)$  it follows that  $M_{\delta_\sigma} = \cup \{M_\alpha : \alpha < \delta_\sigma\} \in N_{\varepsilon+1}$  and  $\bar{M} \restriction \delta_\sigma \in N_{\varepsilon+1} \prec (\mathcal{H}(\chi), \in, <_\chi^*)$  hence

$$(*)_{4.4} \quad M_{\delta_\sigma} \cap N_{\varepsilon+1} \subseteq M_{\sup(N_{\varepsilon+1} \cap \delta_{i(\partial)})}$$

but (by  $(*)_{4.2} + (*)_{4.3}$ )

$$(*)_{4.5} \quad \{g(\zeta) : \zeta \in a\} \in M_{\delta_\sigma} \cap N_{\varepsilon+1}.$$

Now as  $\bar{M}, \bar{\delta} \in N_0$  and  $\sigma \in N_0$  by  $\odot_3(e)$ , clearly  $M_{\delta_\sigma} \in N_0$  and as  $N_{\varepsilon+1} \cap \sigma = i(\varepsilon+1)$  by the choice of  $i(\varepsilon+1)$  after  $\odot_3$  and  $\|N_{\varepsilon+1}\| < \sigma$  by  $\odot_3(b)$  clearly

$$(*)_{4.6} \quad N_{\varepsilon+1} \cap M_{\delta_\sigma} \subseteq M_{\delta_{i(\varepsilon+1)}}.$$

But  $f(\varepsilon+1) = \delta_{i(\varepsilon+1)}$  by  $\oplus_1(e)$  hence by  $(*)_{4.5} + (*)_{4.6}$  we have

$$(*)_{4.7} \quad \{g(\zeta) : \zeta \in a\} \in M_{f(\varepsilon+1)}.$$

So we have proved  $\oplus_4$ .

$\oplus_5$  if  $\varepsilon < \partial, a \in \mathcal{P}_\varepsilon^*, a \subseteq e$  and  $\xi \in a \wedge (a \cap \xi \text{ has a last member})$  then  $\{g(\zeta) : \zeta \in a \cap \xi\} \in M_{f(\xi)}$ .

[Why? Let  $\zeta(*) = \max(a \cap \xi)$ , it is well defined by the assumption on  $\xi$ . But  $\bar{\mathcal{P}}^*$  is a  $(\partial, \partial, < \sigma)$ -system by the assumption of clause (c) (so of clause (c)<sup>+</sup>) of  $\boxplus$ , hence by clause (d) of Definition 0.9(1) we have  $a \cap \zeta(*) \in \mathcal{P}_{\zeta(*)}^*$  and, of course,  $a \cap \zeta(*) \subseteq e$  hence we can apply  $\oplus_4$  with  $(\zeta(*), a \cap \zeta(*))$  here standing for  $(\varepsilon, a)$  there, so we can deduce  $\{g(\zeta) : \zeta \in a \cap \zeta(*)\} \in M_{f(\zeta(*)+1)}$ . But  $\zeta(*)+1 \leq \xi$  hence  $f(\zeta(*)+1) \leq f(\xi)$  hence  $M_{f(\zeta(*)+1)} \subseteq M_{f(\xi)}$ . So  $\{g(\zeta) : \zeta \in a \cap \zeta(*)\} \in M_{f(\xi)}$ , hence by the obvious closure properties of  $M_{f(\xi)} \cap [f(\xi)]^{\leq \theta}$  also  $\{g(\zeta) : \zeta \in a \cap \xi\} \in M_{f(\xi)}$ .]

$\oplus_6$  if  $\varepsilon < \partial, a \in \mathcal{P}_\varepsilon^*$  and  $a \subseteq e$  then the set  $b = \{f(\zeta) : \zeta \in a \text{ and } \text{otp}(a \cap \zeta) \text{ is a successor ordinal}\}$  belongs to  $\mathcal{P}_{f(\varepsilon+1)}$ .

[Why? By  $\oplus_4 + \oplus_5$ , the definition of  $\mathcal{P}_{f(\varepsilon+1)}$  in  $\odot_2$  and the obvious closure properties of each  $M_\alpha$ .]

So we are done proving clause (c)( $\gamma$ ) of  $\boxplus$  hence clause (c). Clause (c)<sup>+</sup>( $\gamma$ )<sup>+</sup> is proved similarly.

We are left with proving clause (d) of  $\boxplus$ , let  $x = \{\lambda, \sigma, \partial, \theta, \bar{\mathcal{P}}^*, E, \bar{M}\}$  and let  $S_1 = \{j \in S_\partial^\sigma : \text{there is } \bar{N} \text{ as in } \odot_3 \text{ such that } j = \sup(\cup\{N_\varepsilon : \varepsilon < \partial\} \cap \sigma)\}$ . Now by the definition 0.13(2) of  $\check{I}_\partial^{\text{ac}}\langle\sigma, \sigma\rangle$  we know that  $S_\partial^\sigma \setminus S_1 \in \check{I}_\partial^{\text{ac}}\langle\sigma, \sigma\rangle$ .

Next, for each  $j \in S_1$  let  $\langle N_\varepsilon : \varepsilon < \partial \rangle$  witness that  $j \in S_1$ . Now choose  $C, g, e, f$  as in  $\oplus_1$ . So by the definition of  $I_\theta^{\text{dg}}(\bar{\mathcal{P}}^*)$  the set  $S_\partial^\partial \setminus S_2 \in I_\theta^{\text{dg}}(\bar{\mathcal{P}}^*)$  where  $S_2 = \{\zeta \in S_\partial^\partial : \text{there is } a \in \mathcal{P}_{<\partial}^* \text{ such that } \text{otp}(a) = \theta, \sup(a) = \zeta \text{ and } a \subseteq e \text{ hence } \zeta \in e\}$ .

For each  $\zeta \in S$ , let  $a \in \mathcal{P}_{<\partial}^*$  witness  $\zeta \in S_2$ , as in the proof of clause (c)( $\gamma$ ) we get that  $\zeta \in \text{good}_\theta''(\bar{\mathcal{P}})$ . Clearly this suffices for proving clauses (d)( $\delta$ ), ( $\varepsilon$ ).  $\square_{1.1}$

**Claim 1.3.** Let  $\sigma > \partial > \theta$ .

- 1)  $S_\partial^\sigma \notin \check{I}_\partial^{\text{ac}}\langle\sigma, \sigma\rangle$  moreover  $\check{I}_\partial^{\text{ac}}\langle\sigma, \sigma\rangle$  is a normal ideal on  $S_\partial^\sigma$ .
- 2) If  $S_1 \in \check{I}_\theta[\sigma]$  and  $S_2 \in \check{I}_\theta^{\text{ac}}\langle\sigma, \partial\rangle$  then  $S_1 \setminus S_2$  is non-stationary.

*Remark 1.4.* If  $\sigma = \partial^+$ , see 0.16.

*Proof.* 1) Easy.

2) Let  $\bar{\mathcal{P}}' = \langle \mathcal{P}'_\varepsilon : \varepsilon < \sigma \rangle$  be a  $(\sigma, \partial, < \sigma)$ -system witnessing  $S_1 \in \check{I}_\theta[\sigma]$ .

Now instead of choosing  $N_\varepsilon$  for  $\varepsilon \leq \partial$  we choose  $N_\varepsilon$  and  $\bar{N}_\varepsilon$  by induction on  $i < \sigma$  such that:

- $\oplus(A)$  (a)  $N_\varepsilon \prec (\mathcal{H}(\chi), \in, <_\chi^*)$
- (b)  $\|N_\varepsilon\| < \sigma$  and  $N_\varepsilon \cap \sigma \in \sigma$
- (c)  $\langle N_\zeta : \zeta \leq \xi \rangle \in N_\varepsilon$  for  $\xi < \varepsilon$
- $(B)$  (a)  $\bar{N}_\varepsilon = \langle N_{\varepsilon,a} : a \in \mathcal{P}'_\varepsilon \rangle$
- (b)  $N_{\varepsilon,a} \prec (\mathcal{H}(\chi), \in, <_\chi^*)$
- (c)  $\|N_{\varepsilon,a}\| < \partial$
- (d) if  $a \in \mathcal{P}'_\varepsilon$  then  $\langle N_{\xi, a \cap \xi} : \xi \in a \cup \{\varepsilon\} \rangle$  is  $\prec$ -increasing and  $\xi \in a \cup \{\zeta\} \wedge \xi = \sup(a \cap \xi) \Rightarrow N_{\xi, a \cap \xi} = \cup \{N_{\zeta, a \cap \zeta} : \zeta \in a\}$
- (e)  $E, \bar{M}, \bar{\delta}, \sigma, \bar{\mathcal{P}}^*$  and  $\bar{\mathcal{P}}'$  belongs to  $N_{i,a}$

- (f)  $\langle N_{\zeta,b} : \zeta \leq \xi, b \in \mathcal{P}'_{\zeta} \rangle$  and  $\langle N_{\zeta} : \zeta \leq \xi \rangle$  belongs to  $N_{\varepsilon,a}$  and to  $N_{\varepsilon}$   
 when  $\xi < \varepsilon_* < \sigma$
- (g)  $\partial + 1 \subseteq N_{\varepsilon,a}$ .

The rest should be clear.  $\square_{1.3}$

*Proof. Proof of 0.1* 1) As  $\partial, \theta$  are regular cardinals and  $\partial > \theta^+$  let  $\bar{\mathcal{P}}^* := \langle \mathcal{P}_{\alpha}^* : \alpha < \partial \rangle$  be a  $(\partial, \leq \theta, < \partial)$ -system satisfying  $S_{\theta}^{\sigma} \notin I_{\theta}^{\text{cg}}(\bar{\mathcal{P}}^*)$ , see 0.13(1), 0.16(3). Let  $\chi, M_*$  be as in 1.1 for our  $\lambda$  such that  $\bar{\mathcal{P}}^* \in M_*$ . Let  $E, \bar{\mathcal{P}}$  be as constructed in 1.1 for our  $\lambda, M_*$  and recall  $\alpha \in \text{nacc}(E) \Rightarrow \text{cf}(\alpha) = \aleph_0$ . So if  $\delta \in E \cap S_{\sigma}^{\lambda}$  then  $\delta \in \text{acc}(E)$  and so there is an increasing continuous sequence  $\langle \delta_i : i < \sigma \rangle$  of members of  $E$  with limit  $\delta$ ; hence by clauses  $(c)^+(\gamma)$  we have  $(\exists^{\text{stat}} i < \delta)[i \in \text{good}_{\theta}''(\bar{\mathcal{P}})]$ .

As we have started with any  $\delta \in E \cap S_{\theta}^{\lambda}$  clearly  $\text{good}_{\theta}''(\bar{\mathcal{P}})$  reflects in any  $\delta \in E \cap S_{\sigma}^{\lambda}$ , but  $\text{good}_{\theta}''(\bar{\mathcal{P}}) \in \check{I}_{\theta}[\lambda]$ . Now by  $\boxplus(b)$  of 1.1  $\delta \in S_{\theta}^{\lambda} \setminus E \Rightarrow \delta \in \text{good}_{\theta}''(\bar{\mathcal{P}})$  so  $\text{good}_{\theta}''(\bar{\mathcal{P}}) \in \check{I}_{\theta}[\lambda]$  is as required.

2) Same proof.

3) Similarly using clause  $(d)(\varepsilon)$  of 1.1.  $\square_{0.1}$

*Proof. Proof of 0.2:*

1) Let  $\chi, \lambda, M_*$  be as the assumption of 1.1 such that in addition  $2^{\theta^{+n}} < M_* \cap \lambda$  for every  $n$ . Let  $E$  and  $\bar{\mathcal{P}} = \langle \mathcal{P}_{\alpha} : \alpha < \lambda \rangle$  be as in the conclusion of 1.1.

Recalling Definition 0.12(2A), let  $S_* = \text{good}_{\theta}''(\bar{\mathcal{P}}) \subseteq S_{\theta}^{\lambda}$ , so obviously  $S_* \in \check{I}_{\theta}[\lambda]$  and for every  $n$  let  $S_n = \{\delta : \text{cf}(\delta) = \theta^{+n} \text{ and } n = 0 \Rightarrow \delta \notin S_* \text{ and } [n \geq 1 \Rightarrow \delta \cap S_{\theta}^{\lambda} \setminus S_* \text{ is a stationary subset of } \delta]\}$ .

Note that by the assumption of part of the theorem

$\boxplus_1$   $S_0$  is a stationary subset of  $\lambda$ .

For  $n \geq 1$  and  $\delta \in S_n$  we choose  $\langle \gamma_{\delta,\varepsilon} : \varepsilon < \text{cf}(\delta) \rangle$ , an increasing continuous sequence with limit  $\delta$  and let  $s_{\delta} = \{\varepsilon < \text{cf}(\delta) : \text{cf}(\varepsilon) = \theta \text{ and } \gamma_{\delta,\varepsilon} \notin S_*\}$ , so as  $\delta \in S_n$  necessary  $s_{\delta}$  is a stationary subset of  $\theta^{+n}$ .

For every stationary  $s \subseteq S_{\theta}^{\theta^{+n}}$  let  $S_{n,s} = \{\delta \in S_n : s_{\delta} = s\}$ , the sequence  $\langle S_{n,s} : s \subseteq S_{\theta}^{\theta^{+n}} \text{ is stationary} \rangle$  is a partition of  $S_n$  and for some club  $E_{n,s} \subseteq E$  of  $\lambda$  we have  $[S_{n,s} \cap E_{n,s} = \emptyset \Leftrightarrow S_{n,s} \text{ is not stationary}]$  for every such  $s$ .

Let  $E_* = \cap \{E_{n,s} : n \geq 1 \text{ and } s \subseteq \theta^{+n} \text{ is stationary}\}$ , so as we are assuming  $2^{\theta^{+n}} < \lambda$ , clearly  $E_*$  is a club of  $\lambda$ .

Clearly if “ $n \geq 1 \wedge (s \subseteq S_{\theta}^{\theta^{+n}} \text{ stationary}) \Rightarrow S_{n,s} \subseteq \lambda$  is not stationary” then  $n = 0, S = S_0$  satisfy the desired conclusion. So assume that  $n \geq 1$  and  $s \subseteq \theta^{+n}$  is stationary and  $S_{n,s}$  is stationary. If  $S_{n,s}$  reflects in no  $S_{\theta+m}^{\lambda}, m > n$  we are done, and also if  $\text{refl}(S_{n,s}) \cap S_{\theta+n+1}^{\lambda}$  reflect in no  $S_{\theta+n}^{\lambda}, m > n+1$ , we are done.

Hence it suffices to prove

$\boxplus_2$  if  $n \geq 1, s \subseteq S_{\theta}^{\theta^{+n}}$  is stationary and  $S_{n,s} \subseteq \lambda$  is stationary,  $n \geq 2, m \geq n+2$  then  $S_{n,s}$  does not reflect in any  $\delta_* \in S_{\theta+m}^{\lambda} \cap \text{acc}(E_*)$ .

Toward this let  $\sigma = \theta^{+m}$  and  $\bar{\delta} = \langle \delta_i : i < \sigma \rangle$  be an increasing continuous sequence of ordinals from  $E_*$  with limit  $\delta_{i(\sigma)} := \delta_*$ . As  $s \subseteq S_{\theta}^{\theta^{+n}}$  is stationary and  $n \geq 2$ , let  $\partial = \theta^{+n}$  by 0.11, 0.16(3) there is  $\bar{\mathcal{P}}^* = \langle \mathcal{P}_{\zeta}^* : \zeta < \partial \rangle$  a  $(\partial, \theta)$ -system such that  $S_{\theta}^{\lambda} \notin I_{\theta}^{\text{cg}}(\bar{\mathcal{P}}^*)$ .

Note that  $\bar{\mathcal{P}}^* \in M_*$  because  $2^{\theta^{+n}} < \lambda$  and  $M_* \cap \lambda$ . So our  $\bar{\mathcal{P}}$  satisfies the conclusion of 1.1, so  $\boxplus$  holds indeed hence we are done.

2),3) The proof is really included in the proof of part (1).  $\square_{0.2}$

*Remark 1.5.* In the proof of 1.1, for regular  $\kappa \in (\theta, \lambda)$  and  $s$  a stationary subset of  $S_\theta^\kappa$  we can let  $S_{\kappa,s} = \{\delta \in S_\kappa^\lambda : \text{for some increasing continuous sequence } \langle \alpha_i : i < \kappa \rangle \text{ of ordinals with limit } \delta, \text{ the set } \{i \in S_\theta^\kappa : i \in s \text{ iff } \alpha_i \in S_*\} \text{ is not stationary}\}$ . Let  $E_{\kappa,s}$  be a club of  $\lambda$ , disjoint to  $S_{\kappa,s}$  if  $S_{\kappa,s}$  is not stationary. Let  $\kappa_* < \lambda$  and  $E_* = \cap \{E_{\kappa,s} : \kappa \in (\theta, \kappa_*) \text{ is regular and } s \subseteq \kappa\}$ . We can then continue as above.

### § 1(B). Quite free witnesses of pcf-cases exist.

**Definition 1.6.** 1) We say  $(\lambda, \bar{\lambda}, J, \kappa)$  is a pcf-case (may omit  $J$  when  $J = [\kappa]^{<\kappa}$  when :

- (a)  $\bar{\lambda} = \langle \lambda_i : i < \kappa \rangle$  is a sequence of regular cardinals  $> \kappa$
- (b)  $J$  is an ideal on  $\kappa$
- (c)  $\lambda = \text{tcf}(\prod_{i < \kappa} \lambda_i, <_J)$ .

2) We say  $\bar{f}$  witness a pcf-case  $(\lambda, \bar{\lambda}, J, \kappa)$  or is a witness for it when  $\bar{f}$  is  $<_J$ -increasing and  $<_J$ -cofinal in  $(\prod_{i < \kappa} \lambda_i, <_J)$ .

3) We say  $\bar{f}$  obeys  $(\lambda, \bar{\lambda}, J, \bar{\mathcal{P}}, \kappa)$  when for some  $\bar{g}, \bar{f}$  obeys  $(\lambda, \bar{\lambda}, J, \kappa, \bar{\mathcal{P}})$  as witnessed by  $\bar{g}$ , see part (4) below and  $\bar{f}$  witnesses the pcf-case  $(\lambda, \bar{\lambda}, J, \kappa)$ . Not mentioning  $\bar{g}$  means for some  $\bar{g}$ .

4) We say that  $\bar{f}$  obeys  $(\lambda, \bar{\mu}, J, \kappa, \bar{\mathcal{P}})$  as witnessed by  $\bar{g}$  when :

- (a)  $\bar{f} = \langle f_\alpha : \alpha < \lambda \rangle$ ;
- (b)  $J$  is an ideal on  $\kappa$  and  $\bar{\mu} = \langle \mu_i : i < \kappa \rangle$
- (c)  $f_\alpha \in {}^\kappa \text{Ord}$
- (d)  $\bar{f}$  is  $<_J$ -increasing
- (e)  $\bar{\mathcal{P}} = \langle \mathcal{P}_\alpha : \alpha < \lambda \rangle$  is a  $(\lambda, \lambda, \leq 2^\lambda)$ -system, so without loss of generality  $\subseteq$ -increasing
- (f)  $\bar{g} = \langle g_a : a \in \bigcup_\alpha \mathcal{P}_\alpha \rangle$
- (g)  $g_a \in {}^\kappa \text{Ord}$
- (h)  $g_a(i) < g_b(i)$  when  $a \triangleleft b$  are from  $\mathcal{P}_{<\lambda}$  and  $|b| < \mu_i$  where  $\mathcal{P}_{<\alpha} := \cup \{ \mathcal{P}_\beta : \beta < \alpha \}$
- (i) if  $a \in \mathcal{P}_\alpha$  then  $g_a <_J f_\alpha$
- (j) if  $\beta \in a \in \mathcal{P}_\alpha, i < \kappa$  and  $|a| < \mu_i$  then  $f_\beta(i) < g_a(i)$ .

**Convention 1.7.** We may allow  $\bar{f} = \langle f_\alpha : \alpha \in S \rangle$  where  $S \subseteq \lambda = \sup(S)$ , that is, say  $\bar{f}$  obeys  $(\lambda, \bar{\mu}, J, \kappa, \bar{\mathcal{P}})$  as witnessed by some  $\bar{g}$  when  $\langle f'_\alpha : \alpha < \lambda \rangle$  satisfies the demands there where  $\alpha \in S \Rightarrow f'_{\text{otp}(S \cap \alpha)} = f_\alpha$ .

**Claim 1.8.** Assume  $(\lambda, \bar{\lambda}, J, \kappa)$  is a pcf-case,  $\mu = \liminf_J(\bar{\lambda})$  and  $\bar{\mathcal{P}}$  is a  $(\lambda, \mu, < \lambda)$ -system.

- 1) There is  $\bar{f}$  obeying  $(\lambda, \bar{\lambda}, J, \kappa, \bar{\mathcal{P}})$ .
- 2) For every  $\bar{f}$  witnessing  $(\lambda, \bar{\lambda}, J, \kappa)$ , for some unbounded  $S \subseteq \lambda$ ,  $\bar{f} \upharpoonright S$  obeys  $(\lambda, \bar{\lambda}, J, \kappa, \bar{\mathcal{P}})$ .
- 3) If  $\bar{f}$  obeys  $(\lambda, \bar{\lambda}, J, \kappa, \bar{\mathcal{P}})$  and  $\theta = \text{cf}(\theta) < \liminf_J(\bar{\lambda})$  then  $S_{\text{gd}}[\bar{f}] \supseteq \text{good}''_\theta(\bar{\mathcal{P}})$ .

*Remark 1.9.* The proof is like the ones in [Sh:e, Ch.I], [Sh:509].

*Proof.* 1) Follows by (2).

2) Let  $\bar{f} = \langle f_\alpha : \alpha < \lambda \rangle$  witness the pcf-case  $(\lambda, \bar{\lambda}, J, \kappa)$ .

By induction on  $\beta < \lambda$  we choose  $\langle g_a : a \in \mathcal{P}_\beta \rangle$  and  $\alpha(\beta)$  such that

- ⊞ (a)  $g_a \in \Pi \bar{\lambda}$
- (b) if  $i < \kappa, b \triangleleft a$  and  $\{a, b\} \subseteq \mathcal{P}_{<\beta}$  and  $|a| < \lambda_i$  then  $g_b(i) < g_a(i)$
- (c)  $\alpha(\beta) < \lambda$  and  $\beta_1 < \beta \Rightarrow \alpha(\beta_1) < \alpha(\beta)$
- (d) if  $i < \kappa, \beta_1 \in a \in \mathcal{P}_\beta$  and  $|a| < \lambda_i$  then  $f_{\alpha(\beta_1)}(i) < g_a(i)$
- (e) if  $a \in \mathcal{P}_{\leq \beta}$  then  $g_a <_J f_{\alpha(\beta)}$ .

In stage  $\beta$  we first choose  $g_a$  for  $a \in \mathcal{P}_\beta \setminus \mathcal{P}_{<\beta}$ , note that this means that for every  $i < \kappa$ , we have to choose  $g_a(i)$  as an ordinal  $< \lambda_i$ , which is a regular cardinal and if  $|a| < \lambda_i$  it should be bigger than  $\leq |a|$  ordinals  $< \lambda_i$ , so this is easy.

As for  $\alpha(\beta)$  for each  $a \in \mathcal{P}_{\leq \beta}$ , as  $\bar{f}$  is cofinal in  $(\Pi \bar{\lambda}, <_J)$  there is  $\gamma_a < \lambda$  such that  $g_a <_J f_{\gamma_a}$ . So  $\alpha(\beta)$  should be an ordinal  $< \lambda$  and  $> \sup\{\alpha(\beta_1); \beta_1 < \beta\}$  which is an ordinal  $< \lambda$ , as  $\lambda$  is regular and it also should be  $> \sup\{\gamma_a : a \in \mathcal{P}_{\leq \beta}\}$  which is  $< \lambda$  as  $\lambda$  is regular  $> |\mathcal{P}_\alpha|$ .

3) Straight.

□<sub>1.8</sub>

**Definition 1.10.** Let  $J$  be an ideal on  $\kappa$ , we may omit it below when  $J = J_\kappa^{\text{bd}}$ .

- 1) A set  $\mathcal{F} \subseteq {}^\kappa \text{Ord}$  is  $J$ -free when there is a sequence  $\langle a_f : f \in \mathcal{F} \rangle$  of members of  $J$  such that  $f_1 \neq f_2 \wedge \{f_1, f_2\} \subseteq \mathcal{F} \wedge i \in \kappa \setminus a_{f_1} \setminus a_{f_2} \Rightarrow f_1(i) \neq f_2(i)$ .
- 2) A set  $\mathcal{F} \subseteq {}^\kappa \text{Ord}$  is  $(\theta, J)$ -free when  $\mathcal{F}'$  is  $J$ -free whenever  $\mathcal{F}' \subseteq \mathcal{F}$  has cardinality  $< \theta$ .
- 3) A sequence  $\langle f_\alpha : \alpha < \alpha_* \rangle$  of members of  ${}^\kappa \text{Ord}$  is a  $(\theta, J)$ -free sequence when, for every  $u \in [\alpha_*]^{<\theta}$  there is a sequence  $\langle a_\alpha : \alpha \in u \rangle$  of members of  $J$  such that: if  $\alpha < \beta$  are from  $u$  then  $i \in \kappa \setminus a_\alpha \setminus a_\beta \Rightarrow f_\alpha(i) < f_\beta(i)$ .
- 4) A set  $\mathcal{F} \subseteq {}^\kappa \text{Ord}$  (we may use a sequence listing it) is called  $(\theta_2, \theta_1, J)$ -free when for every  $\mathcal{F}' \subseteq \mathcal{F}$  of cardinality  $< \theta_2$ , we can find a partition  $\langle \mathcal{F}'_\varepsilon : \varepsilon < \varepsilon(*) \rangle$  of  $\mathcal{F}'$  such that:

- each  $\mathcal{F}'_\varepsilon$  has cardinality  $< \theta_1$
- we can find a sequence  $\langle s_f : f \in \mathcal{F}' \rangle$  of members of  $J$  such that  $f_1 \in \mathcal{F}'_{\varepsilon_1} \wedge f_2 \in \mathcal{F}'_{\varepsilon_2} \wedge \varepsilon_1 \neq \varepsilon_2 \wedge i \in \kappa \setminus s_{f_1} \setminus s_{f_2} \Rightarrow f_1(i) \neq f_2(i)$ .

4A) A set  $\mathcal{F} \subseteq {}^\kappa \text{Ord}$  is called  $(\theta_2, \theta_1, J)$ -free when for every  $\mathcal{F}' \subseteq \mathcal{F}$  of cardinality  $\theta_2$ , there is a  $J$ -free  $\mathcal{F}'' \subseteq \mathcal{F}'$  of cardinality  $\theta_1$ .

4B) Similarly to 4), 4A) for a sequence  $\langle f_\alpha : \alpha < \alpha_* \rangle$  of members of  ${}^\kappa \text{Ord}$  means that it is with no repetitions and  $\{f_\alpha : \alpha \in u\}$  satisfies the requirement.

5) A set  $\mathcal{F} \subseteq {}^\kappa \text{Ord}$  is called  $(\theta_2, \theta_1, J)$ -stable when for every  $u \subseteq \text{Ord}$  of cardinality  $< \theta_1$  the set  $\{f \in \mathcal{F} : i \text{ the set } \{i < \kappa : f(i) \in u\} \text{ is not in } J\}$  has cardinality  $< \theta_2$ .

5A) A set  $\mathcal{F} \subseteq {}^\kappa \text{Ord}$  is  $(\theta, J)$ -stable when it is  $(\theta, \theta, J)$ -stable.

5B) A set  $\mathcal{F} \subseteq {}^\kappa \text{Ord}$  is  $(\theta_2, \theta_1, J)$ -stable when for every  $\theta \in [\theta_2, \theta_1]$  is  $(\theta, J)$ -stable.



Toward proving Theorem 0.4 we prove

**Claim 1.11.** *If (A) then (B) where:*

- (A) (a)  $(\lambda, \bar{\lambda}, J, \kappa)$  is a pcf-case  
 (b)  $M_* \prec (\mathcal{H}(\lambda^+), \theta, <_{\lambda^+}^*)$  has cardinality  $< \lambda$ ,  $M_* \cap \lambda \in \lambda$  and  $(\lambda, \bar{\lambda}, J, \kappa) \in M_*$ ; (clearly exists and by 1.1, 1.8 there are  $\bar{\mathcal{P}}, E, \bar{f}$ , as required below)  
 (c)  $\bar{f}, \bar{\mathcal{P}}, E$  are as in 1.1 for our  $\lambda, M_*$   
 (c)  $\bar{f}^1$  obeys  $(\lambda, \bar{\lambda}, J, \kappa, \bar{\mathcal{P}})$   
 (d)  $\mu$  is a limit uncountable cardinal  
 (e)  $\mu = \liminf_J(\bar{\lambda})$ , i.e.  $\mu = \min\{\chi: \text{the set } \{i < \kappa : \lambda_i < \chi\} \text{ is not from } J\}$   
 (f)  $\partial = \text{cf}(\partial) < \kappa$ ,  $J$  is  $\partial^+$ -complete  
 (g)  $S \subseteq S_\partial^\lambda$  is stationary such that  $\delta \in S \Rightarrow (\mu^2 \text{ divide } \delta)$   
 (h)  $\bar{\alpha} = \langle \alpha_{\delta,i} : \delta \in S, i < \partial \rangle$  where  $\bar{\alpha}_\delta = \langle \alpha_{\delta,i} : i < \partial \rangle$  is increasing continuous with limit  $\delta$  such that  $\alpha_{\delta,i}$  is divisible by  $\mu$   
 (i)  $\bar{f} = \bar{f}^2 = \langle f_\delta^2 : \delta \in S \rangle$  is where  $f_\delta^2 : \partial \times \kappa \rightarrow \delta$  is defined by  $f_\delta^2(i, j) = \alpha_{\delta,i} + f_\delta^1(j)$   
 (j)  $J_* = J_\partial^{\text{bd}} \times J = \{u \subseteq \partial \times \kappa : \text{for every } i < \partial \text{ large enough, } \{j < \kappa : (i, j) \in u\} \in J\}$ ; of course, we can translate  $J_*$  to an ideal on  $\{v \subseteq \kappa : \{(i, j) \in \partial \times \kappa : \partial \cdot j + i \in v\} \in J_*\}$ .  
 (B) (a)( $\alpha$ ) if  $\theta \in [\kappa, \mu)$  then the sequence  $\bar{f}^2$  is  $(\theta^{+\text{comp}(J)+1}, \theta^{+4}, J_*)$ -free recalling  $\partial < \text{comp}(J) \leq \kappa$ , see 1.13 and 0.8(5)  
 ( $\beta$ )  $\bar{f}^2$  is  $(\text{comp}(J), J_*)$ -free  
 ( $\gamma$ ) if  $\theta \in [\kappa, \mu)$  is a limit cardinal and  $\text{cf}(\theta) \notin [\text{comp}(J), \kappa^+)$  and  $(\forall \Upsilon)(\kappa < \Upsilon < \mu \wedge \text{cf}(\Upsilon) \in [\text{comp}(J), \kappa^+) \Rightarrow \text{pp}_J(\mu) < \theta)$  then  $\bar{f}^2$  is  $(\theta^{+\text{comp}(J)+1}, \theta^+, J_*)$ -free  
 (b) if  $\sigma$  is regular and  $\delta \in S_\sigma^\lambda$  and  $\sigma < \mu$  then, see Definition 0.18:  
 ( $\alpha$ )  $\kappa^{+4} \leq \sigma \Rightarrow \delta \notin S_J^{\text{ch}}[\bar{f}]$   
 ( $\beta$ )  $\kappa^+ < \sigma < \kappa^{+\text{comp}(J)+1} \Rightarrow \delta \notin S_J^{\text{bd}}[\bar{f}]$   
 ( $\gamma$ )  $\kappa \leq \theta \wedge \theta^{+4} \leq \sigma < \theta^{+\text{comp}(J)+1} \Rightarrow \delta \notin S_J^{\text{bd}}[\bar{f}]$ .

*Remark 1.12.* This continues [Sh:108] and [MgSh:204]; note that here  $\partial < \kappa$ . This helps; there are relatives with  $\sigma \geq \kappa$  but not needed at present.

*Proof.* Note that

$$\boxplus_1 \text{ if } \theta = \text{cf}(\theta) \in \mu \setminus \kappa^+ \text{ then } S_{\text{gd}}[\bar{f}] \cap S_\theta^\lambda \supseteq \text{good}_\theta''[\bar{\mathcal{P}}].$$

[Why? By 1.8(3).]

$$\boxplus_2 \text{ if } \theta, \sigma \text{ are regular cardinals from } (\kappa, \mu) \text{ and } \theta^{+2} < \sigma \text{ then } S_{\text{gd}}[\bar{f}] \cap S_\theta^\lambda \text{ reflect in every } \delta \in S_\sigma^\lambda.$$

[Why? Let  $\Upsilon = \theta^{+2}$ , hence by 0.16(3) there is a  $(\Upsilon, \theta, < \Upsilon)$ -system such that  $S_\theta^\Upsilon \notin I_{\theta^{\text{cg}}}^\Upsilon[\Upsilon]$ , see Definition 0.13(1) hence by 1.1, that is the choice of  $\bar{\mathcal{P}}$ , the set  $\text{good}_\theta''(\bar{\mathcal{P}}) \subseteq S_\theta^\lambda$  reflect in every  $\delta \in S_\sigma^\lambda$ , and so by  $\boxplus_1$  we are done.]

$\boxplus_3$   $S_J^{\text{gd}}[\bar{f}]$  include  $\{\delta < \lambda : \theta^{+4} \leq \text{cf}(\delta) < \theta^{+\text{comp}(J)+1}\}$  when  $\theta \in [\kappa, \mu]$ .

[Why? By  $\boxplus_2$ , 0.19(2), 0.20(1),(3).]

So we have proved (b) of (B); concerning  $(B)(b)(\gamma)$  recall that

- if  $\delta \in S_J^{\text{ch}}[\bar{f}]$  then for some club  $e$  of  $\delta$  we have  $\alpha \in e \wedge \text{cf}(\alpha) > \kappa \Rightarrow \alpha \in S_J^{\text{ch}}[\bar{f}]$ , (similarly for  $S_J^{\text{gd}}[\bar{f}]$ )

$\boxplus_4$   $\bar{f}^2$  is  $(\kappa^{+\text{comp}(J)+1}, \kappa^{+4}, J)$ -free, see Definition 1.10(4), that is as a set.

[Why? By  $\boxplus_6$  proved below using  $\boxplus_3$ .]

$\boxplus_5$  if  $\theta \in [\kappa, \mu]$  then  $\bar{f}^2$  is  $(\theta^{+\text{comp}(J)+1}, \theta^{+4}, J)$ -free.

[Why? By  $\boxplus_6$  below using  $\boxplus_3$ .]

$\boxplus_6$  if  $\theta_2 > \theta_1 = \text{cf}(\theta_1) > \kappa$  and  $\delta < \lambda \wedge \theta_1 \leq \text{cf}(\delta) < \theta_2 \Rightarrow \delta \in S_J^{\text{gd}}[\bar{f}]$  then  $\bar{f}^2$  is  $(\theta_2, \theta_1, J_*)$ -free.

Toward this we prove for  $\theta \in [\theta_1, \theta_2)$  that

- $\oplus_{\bar{f}, \theta}$  if  $u \subseteq S$ , recalling  $S \subseteq S_\sigma^\lambda$ ,  $|u| = \theta$  then we can find  $\bar{s} = \langle s_\alpha : \alpha \in u \rangle \in {}^u(J_*)$  such that in the graph  $(u, R_{\bar{s}})$  every node has valency  $< \theta_1$  where:
  - for  $u \subseteq \lambda$  and  $\bar{s} \in {}^u J_*$  let  $(u, R_{\bar{s}})$  be the following graph:  $\alpha R_{\bar{s}} \beta$  iff  $\alpha \neq \beta \in u$  and for some  $(i, j) \in \sigma \times \kappa$ , we have  $(i, j) \notin s_\alpha \cup s_\beta$  and  $f_\alpha(i) = f_\beta(j)$ .

Why this suffice? As then let  $\langle u_t : t \in I \rangle$  list the components of the graph  $(u, R_{\bar{s}})$ , so necessarily each component has cardinality  $< \theta$ , recalling  $\theta_1$  is regular, so  $\langle \{f_\alpha : \alpha \in u_t\} : t \in I \rangle$  is a partition as required in Definition 1.10(4).

Why this is true? We prove this by induction on  $\text{otp}(u)$ .

Case 1:  $\text{otp}(u) < \theta_1$

Let  $s_\alpha = \emptyset \in J_*$  for  $\alpha \in u$ , clearly as required.

Case 2:  $\text{otp}(u) = \zeta + 1$

Let  $\alpha = \max(u)$ , let  $\bar{s} \in {}^{u \cap \alpha}(J_*)$  be as promised for  $u \cap \alpha$  and define  $\bar{s}' \in {}^u(J_*)$  by  $s'_\beta$  is  $s_\beta$  if  $\beta < \alpha$  and is  $\emptyset$  if  $\beta = \alpha$ , now check.

Case 3:  $\delta = \text{otp}(u)$  is a limit ordinal of cofinality  $< \theta_1$

Let  $\sigma := \text{cf}(\delta)$  and  $\langle \alpha_\varepsilon : \varepsilon < \sigma \rangle$  be increasing continuous with limit  $\sup(u)$  such that  $\alpha_0 = 0$ . For  $\varepsilon < \sigma$  let  $u_\varepsilon = u \cap [\alpha_\varepsilon, \alpha_{\varepsilon+1})$  and let  $\bar{s}_\varepsilon = \langle s_\alpha : \alpha \in u_\varepsilon \rangle$  be as required for  $u_\varepsilon$ , exists as  $\text{otp}(u_\varepsilon) < \text{otp}(u)$ . So  $\bar{s} = \langle s_\alpha : \alpha \in u \rangle$  is well defined. Now for each  $\beta \in u$ ,  $(i_*, j_*) \in \partial \times \kappa$  and  $\varepsilon$  the set  $w_{\beta, \varepsilon, i_*, j_*} = \{\gamma \in u_\varepsilon : (i_*, j_*) \notin s_\gamma \text{ and } f_\gamma^2(i_*, j_*) = f_\beta^2(i_*, j_*)\}$  has cardinality  $< \theta_1$  because  $\gamma_1, \gamma_2 \in w_{\beta, \varepsilon, i_*, j_*} \Rightarrow ((i_*, j_*) \in \partial \times \kappa \setminus (s_{\gamma_1} \cup s_{\gamma_2})) \wedge f_{\gamma_1}^2(i_*, j_*) = f_{\gamma_2}^2(i_*, j_*)$ ; hence  $w_\beta := \cup \{w_{\beta, \varepsilon, i, j} : \varepsilon < \text{otp}(u) \text{ and } i < \partial, j < \kappa\}$  has cardinality  $< \theta_1$  and  $\langle w_\beta : \beta \in u \rangle$  is as required.

Case 4:  $\delta = \text{otp}(u)$  has cofinality  $\geq \theta_1$ .

We choose  $\bar{s} \in {}^u(J_*)$ ,  $\bar{\beta}, \bar{a}^1$  such that:

- (\*)<sub>6.1</sub> (a)  $\bar{\beta} = \langle \beta_\varepsilon : \varepsilon < \text{cf}(\delta) \rangle$  is increasing continuous
- (b)  $\beta_0 = 0$
- (c)  $\cup \{\beta_\varepsilon : i < \text{cf}(\delta)\} = \sup(u)$
- (d)  $\bar{a}^1 = \langle a_\varepsilon^1 : \varepsilon < \text{cf}(\delta) \text{ non-limit} \rangle$

- (e)  $a_\varepsilon^1 \in J$
- (f) if  $\varepsilon > 0$  then  $\beta_\varepsilon = \sup(u \cap \beta_\varepsilon)$
- (g) if  $\varepsilon, \zeta < \text{cf}(\delta)$  are non-limit and  $j \in \kappa \setminus a_\varepsilon^1 \setminus a_\zeta^1$  then  $f_{\beta_\varepsilon}^1(j) < f_{\beta_\zeta}^1(j)$
- (h)  $\beta_\varepsilon \in S_\sigma^\lambda$  iff  $\text{cf}(\varepsilon) = \sigma$ .

[Why such  $\bar{\alpha}, \bar{a}$  exist? First,  $\sup(u) \in S_J^{\text{gd}}[\bar{f}^1]$  holds by an assumption of  $\boxplus_6$  because  $\theta_1 \leq \text{cf}(\sup(u))$  by the case assumption and  $\text{cf}(\sup(u)) < \theta_2$  as  $|u| \leq \theta_2$ . Second, use Definition 0.18(1) recalling clause (d) of  $(*)_{6.1}$ .]

$(*)_{6.2}$  we can find  $\bar{a}$  such that:

- (a)  $\bar{a} = \langle a_\varepsilon : \varepsilon < \text{cf}(\delta) \rangle$
- (b)  $a_\varepsilon = a_\varepsilon^1$  if  $\varepsilon$  is non-limit
- (c)  $a_\varepsilon \in J$
- (d) if  $\varepsilon < \zeta < \text{cf}(\delta)$  and  $\text{cf}(\zeta) < \text{comp}(J)$  or  $\text{cf}(\zeta) > \kappa$  then  $j \in \kappa \setminus a_\varepsilon \setminus a_\zeta \Rightarrow f_{\beta_\varepsilon}(j) < f_{\beta_\zeta}(j)$ .

[Why? For non-limit  $\varepsilon < \text{cf}(\delta)$  let  $a_\varepsilon = a_\varepsilon^1$ .

If  $\varepsilon < \text{cf}(\delta)$  and  $\aleph_0 \leq \text{cf}(\varepsilon) < \text{comp}(J)$  then let  $e_\varepsilon$  be an unbounded subset of  $\varepsilon$  of order type  $\text{cf}(\varepsilon)$  and let  $a_\varepsilon = \kappa \setminus \{i < \kappa : i \notin \cup \{a_{\beta_{\zeta+1}} : \zeta \in e_\varepsilon\} \text{ and } f_{\beta_\varepsilon}^1(i) < f_{\beta_{\varepsilon+1}}^1(i) \text{ and } \zeta \in e_\varepsilon \Rightarrow f_{\beta_{\zeta+1}}^1(i) < f_{\beta_\varepsilon}^1(i)\}$ .

As  $J$  is  $\text{comp}(J)$ -complete ideal on  $\kappa$  and  $\bar{f}^1$  is  $<_J$ -increasing clearly  $a_\varepsilon \in J$ .

If  $\varepsilon < \text{cf}(\delta)$  and  $\text{cf}(\varepsilon) > \kappa$  then let  $a_\varepsilon = \{i < \kappa : \text{the set } \{\zeta < \varepsilon : i \notin a_{\zeta+1} \text{ and } f_{\beta_{\zeta+1}}(i) < f_{\beta_\varepsilon}(i)\} \text{ is a bounded subset of } \varepsilon\}$ .

Toward proving  $a_\varepsilon \in J$ , first we find  $\xi(\varepsilon) < \varepsilon$  such that: if  $i < \kappa$  and the set  $\{\zeta < \varepsilon : i \in \kappa \setminus a_{\zeta+1} \text{ and } f_{\beta_{\zeta+1}}(i) < f_{\beta_\zeta}(i)\}$  is bounded below  $\varepsilon$  then it is  $\leq \xi(\varepsilon)$ ; this is possible as  $\text{cf}(\varepsilon) > \kappa$ .

So  $\kappa \setminus a_\varepsilon \supseteq \{i < \kappa : f_{\beta_{\xi(\varepsilon)+1}}^1 < f_{\beta_\varepsilon}^1(i) \text{ and } i \notin a_{\xi(\varepsilon)+1}\}$  and the latter set is  $= \kappa \bmod J$  because  $(a_{\xi(\varepsilon)+1} \in J) \wedge (f_{\beta_{\xi(\varepsilon)+1}} <_J f_{\beta_\varepsilon}^1)$ ; it follows that  $a_\varepsilon \in J$ .

In the remaining cases  $\text{cf}(\varepsilon) \in [\text{comp}(J), \kappa]$  let  $a_\varepsilon = \kappa \setminus \{i < \kappa : f_{\beta_\varepsilon}(i) < f_{\beta_{\varepsilon+1}}(i) \text{ and } i \notin a_{\varepsilon+1}\}$ . Actually only the  $a_\varepsilon$  for  $\varepsilon \in S_\delta^{\text{cf}(\delta)}$  are used later.

Let us check that  $\langle a_\varepsilon : \varepsilon < \text{cf}(\delta) \rangle$  is as required in  $(*)_{6.2}$  so assume  $\varepsilon < \zeta < \text{cf}(\delta)$  and  $i \in \kappa \setminus a_\varepsilon \setminus a_\zeta$ . First, if  $\varepsilon, \zeta$  are non-limit then  $i \in \kappa \setminus a_\varepsilon^1 \setminus a_\zeta^1$  hence  $f_{\beta_\varepsilon}(i) < f_{\beta_\zeta}(i)$ . Second, if  $\varepsilon$  is non-limit and  $\text{cf}(\zeta) < \text{comp}(J)$  then we can find  $\xi \in e_\zeta$  which is  $> \varepsilon$ , so  $i \notin a_{\beta_{\xi+1}}$  as  $a_{\beta_{\xi+1}} \subseteq a_{\beta_\varepsilon}$  hence  $f_{\beta_\varepsilon}(i) < f_{\beta_{\xi+1}}(i)$  and by the choice of  $a_{\alpha_\varepsilon}$  also  $f_{\beta_{\xi+1}}(i) < f_{\beta_\zeta}(i)$ , together  $f_{\beta_\varepsilon}(i) < f_{\beta_\zeta}(i)$ . Third, if  $\varepsilon$  is a limit ordinal and  $\text{cf}(\zeta) < \text{comp}(J)$  so by the choice of  $a_\varepsilon$  we have  $(f_{\beta_\varepsilon}(i) < f_{\beta_{\varepsilon+1}}(i)) \wedge (a_\varepsilon \supseteq a_{\varepsilon+1})$  so  $i \notin a_{\varepsilon+1}$ ; so by the above applied to  $(\varepsilon+1, \zeta)$  we have  $f_{\beta_{\varepsilon+1}}(i) < f_{\beta_\zeta}(i)$ , so together  $f_{\beta_\varepsilon}(i) < f_{\beta_\zeta}(i)$ . The cases when  $\text{cf}(\zeta) > \kappa$  is similar. So we have proved  $(*)_{6.2}$ .]

Now for each  $\varepsilon < \text{cf}(\delta)$  let  $u_\varepsilon = u \cap [\beta_\varepsilon, \beta_{\varepsilon+1})$  hence  $\text{otp}(u_\varepsilon) < \text{otp}(u) = \delta$  hence there is a sequence  $\langle s_\alpha^\varepsilon : \alpha \in u_\varepsilon \rangle$  of members of  $J_*$  as required. For each  $\varepsilon < \text{cf}(\delta)$  and  $\beta \in u_\varepsilon \setminus \{\beta_\varepsilon\}$  let  $\mathbf{i}(\beta) < \partial$  be such that  $\{\alpha_{\beta,i} : i \in [\mathbf{i}(\beta), \sigma)\} \cap \beta_\varepsilon = \emptyset$  and if  $\varepsilon < \text{cf}(\delta)$ ,  $\beta = \beta_\varepsilon$  so  $\beta_\varepsilon \in S_\sigma^\lambda$  let  $\mathbf{i}(\alpha) = 0$ .

Lastly, let us define  $\bar{s} = \langle s_\beta : \beta \in u \rangle$ :

- $(*)$  if  $\beta \in u_\varepsilon$  then  $s_\beta := s_\beta^\varepsilon \cup \{(i, j) \in \partial \times \kappa : i \leq \mathbf{i}(\beta)\} \cup \{(i, j) \in \partial \times \kappa : j \in a_\varepsilon \cup a_{\varepsilon+1}\} \cup \{(i, j) \in \partial \times \kappa : \neg(f_{\beta_\varepsilon}^1(j) \leq f_\beta^1(j) < f_{\beta_{\varepsilon+1}}^1(j))\}$ .

Let  $\beta \in u$  and let  $w_\beta = \{\gamma \in u : \text{there is } (i, j) \in \sigma \times \kappa \setminus s_\beta \setminus s_\gamma \text{ satisfying } f_\gamma(i, j) = f_\beta(i, j)\}$  and we have to prove that  $w$  has cardinality  $< \theta_1$ . Let  $\varepsilon < \text{cf}(\delta)$  be such that  $\beta \in u_\varepsilon$  that is  $\beta \in [\beta_\varepsilon, \beta_{\varepsilon+1})$ , clearly  $\varepsilon$  exists and is unique. As  $s_\beta \supseteq s_\beta^\varepsilon$  clearly  $w_\beta \cap [\beta_\varepsilon, \beta_{\varepsilon+1})$  have cardinality  $< \theta_1$ . Now if  $\gamma \in u \cap \beta_\varepsilon \wedge \beta > \beta_\varepsilon$  then by the choice of  $s_\beta$  we have  $s_\beta \supseteq \mathbf{i}(\beta) \times \kappa$  and by the choice of  $\mathbf{i}(\beta)$  we have  $\gamma \notin w_\beta$  recalling  $\{\alpha_{\gamma,j} : j < \partial\} \subseteq \beta_\varepsilon$ . If  $\gamma \in u \cap \beta_\varepsilon \wedge \beta = \beta_\varepsilon$  then necessarily  $\beta_\varepsilon \in S_\partial^\lambda$  so  $\text{cf}(\beta_\varepsilon) = \partial$  and let  $\xi < \text{cf}(\delta)$  be such that  $\gamma \in [\beta_\xi, \beta_{\xi+1})$ , now if  $(i, j) \in \partial \times \kappa \setminus s_\beta \setminus s_\gamma$  then by  $(*)_{6.2(d)}$  we have  $f_\gamma^1(i) < f_{\alpha_{\xi+1}}^1(i) < f_{\alpha_\xi}^1(i)$  so  $\gamma \notin w_\varepsilon$ . Together  $w_\beta \cap \alpha_\varepsilon = \emptyset$ .

Next, assume  $\gamma \in u \setminus \beta_{\varepsilon+1}$  say  $\gamma \in u_\xi, \xi > \varepsilon$ ; if  $\text{cf}(\xi) \neq \partial \vee \gamma > \beta_\xi$  we use  $\mathbf{i}(\gamma) \times \kappa \subseteq s_\gamma$  and if  $\text{cf}(\xi) = \partial \wedge \gamma = \beta_\xi$  we use the choices of  $a_\xi, a_\varepsilon$ ; hence  $w_\beta \setminus \beta_{\varepsilon+1} = \emptyset$ .

Together  $w_\beta$  has cardinality  $< \theta_1$  as required. So we are done proving Case 4, hence proving  $\boxplus_6$ .

$\boxplus_7$  the sequence  $\bar{f}^2$  is  $(\text{comp}(J)^+, J_*)$ -free; this is clause (a)( $\beta$ ) of (B).

[Why? Let  $u \subseteq \lambda$  have cardinality  $\leq \text{comp}(J)$ , let  $\langle \beta_\varepsilon : \varepsilon < |u| \rangle$  list  $u$  and  $a_\varepsilon = \{i < \kappa : \text{for some } \zeta < \varepsilon \text{ we have } f_{\beta_\zeta}^1(i) = f_{\beta_\varepsilon}^1(i)\}$ , so as  $J$  is  $|u|^+$ -complete by the assumption clearly  $a_\varepsilon \in J$ . Let  $s_{\beta_\varepsilon} = \partial \times a_\varepsilon$  for  $\varepsilon < |u|$ , recalls that for each  $\zeta < \varepsilon, \{i < \kappa : f_{\beta_\zeta}^1(i) = f_{\beta_\varepsilon}^1(i)\} \in J$  by clause (A)(c) of the assumption and so  $\langle s_\beta : \beta \in u \rangle$  is as required.]

$\boxplus_8$  if  $\theta \in [\kappa, \mu)$  then  $\bar{f}^2$  is  $(\theta^{+\text{comp}(J)+1}, \theta^{+4}, J_*)$ -free.

[Why? By  $\boxplus_6$  and (B)(b)( $\gamma$ ) which we have proved in  $\boxplus_3$ .]

$\boxplus_9$  if  $\theta \in [\kappa, \mu)$  is a limit cardinal and  $\text{cf}(\theta) \notin [\text{comp}(J), \kappa^+)$  and  $(\forall \Upsilon)(\kappa < \Upsilon < \mu \wedge \text{cf}(\Upsilon) \in [\text{comp}(J), \kappa^+) \Rightarrow \text{pp}_J(\mu) < \theta)$  then  $\bar{f}^2$  is  $(\theta^{+\text{comp}(J)+1}, \theta, J_*)$ -free. This is clause (B)(a)( $\gamma$ ) of the desired conclusion.

Why? Clearly  $\theta \neq \kappa$  hence recalling  $\theta$  is a limit ordinal  $\geq \kappa$  we have  $\theta \geq \kappa^{+4}$ . Again by  $\boxplus_6$  it suffices to prove that if  $\delta < \lambda$  and  $\text{cf}(\delta) \in [\theta, \theta^{+\text{comp}(J)+1})$  then  $\delta \notin S_J^{\text{ch}}[\bar{f}]$  and  $\delta \notin S_J^{\text{bd}}[\bar{f}]$ .

If  $\text{cf}(\delta) \geq \theta^{+4}$  this holds by  $\boxplus_3$ , so we can assume  $\text{cf}(\delta) \in \{\theta^{+\ell} : \ell \leq 3\}$ . Now  $\delta \notin S_J^{\text{ch}}[\bar{f}]$  as otherwise there is a club  $e$  of  $\delta$  such that  $\alpha \in e \wedge \text{cf}(\alpha) > \kappa \Rightarrow \alpha \in S_J^{\text{ch}}[\bar{f}]$ , contradicting  $\boxplus_3$  applied to  $\kappa^{+4}$ .

Also  $\delta \notin S_J^{\text{bd}}[\bar{f}]$  as otherwise  $\text{cf}(\delta) = (\prod_{i < \kappa} \sigma_i, <_J)$  for some  $\sigma_i = \text{cf}(\sigma_i) \in (\kappa, \mu)$

but this contradicts the assumption of  $\boxplus_9$ , e.g. (B)(b)( $\gamma$ ).  $\square_{1.11}$

*Proof. Proof of 0.4:*

The proof is by cases.

Case 1:  $\lambda$  is singular.

In this case there is a  $\mu^+$ -free  $\mathcal{F} \subseteq {}^\kappa\mu$  of cardinality  $2^\mu = \lambda$  by [Sh:898, 3.10(3)=1f.28(3)]; more fully by [Sh:g, Ch.II,2.3,pg.53] for every  $\chi \in (\mu, \lambda)$  there is a  $\mu^+$ -free  $\mathcal{F}_\chi \subseteq {}^\kappa\mu$  of cardinality  $\chi$ ; by letting  $\bar{\chi} = \langle \chi_\varepsilon : \varepsilon < \text{cf}(\lambda) \rangle$  be increasing with limit  $\lambda$ , combining the  $\mathcal{F}_{\chi_\varepsilon}$ 's and  $\mathcal{F}_{\text{cf}(\lambda)}$  we are done. So clause (A) holds and we are done.

Case 2:  $\lambda$  is regular and  $|\alpha|^{<\kappa} = \lambda$  for some  $\alpha < \lambda$ .

In this case by [Sh:898, 3.6=1f.21] there is a  $\mu^+$ -free  $\mathcal{F} \subseteq {}^\kappa\mu$  of cardinality  $2^\mu = \lambda$  so again clause (A) holds and we are done.

**Case 3:**  $\lambda$  is regular and  $\alpha < \lambda \Rightarrow |\alpha|^{<\sigma} < \lambda$ .

Let  $E = \{\delta < \lambda : \alpha < \lambda \Rightarrow |\alpha|^{<\kappa} < \delta \text{ and } \delta \text{ is divisible by } \mu \cdot \mu\}$ , clearly a club of  $\lambda$ .

Let  $S \subseteq E$  be any stationary subset of  $S_\sigma^\lambda$ . We choose  $\langle \bar{\alpha}_\delta : \delta \in S \rangle$  such that  $\bar{\alpha}_\delta = \langle \alpha_{\delta,i} : i < \sigma \rangle$  is increasing with limit  $\delta$  such that each  $\alpha_{\delta,i}$  is divisible by  $\mu$ . By the case assumption we have  $S \in \check{I}_\sigma[\lambda]$ , hence without loss of generality  $\alpha_{\delta_1,i_1} = \alpha_{\delta_2,i_2} \Rightarrow i_1 = i_2 \wedge (\forall i < i_1)(\alpha_{\delta_1,i} = \alpha_{\delta_2,i})$ .

Now as  $\mu \in \mathbf{C}_\kappa$ , there is a sequence  $\bar{\lambda}$  such that  $(\lambda, \bar{\lambda}, J_\kappa^{\text{bd}}, \kappa)$  is a pcf-case such that  $\bar{\lambda}$  is an increasing sequence of regular cardinals with limit  $\mu$ . We can choose  $\chi, M_*$  as in the assumption of 1.1 for  $\lambda$  such that  $\mathcal{H}(\mu) \in M_*$  and then choose  $E, \bar{\mathcal{P}}$  as in the conclusion of 1.1.

Hence by 1.8(1) we can find  $\bar{f}^1 = \langle f_\alpha^1 : \alpha < \lambda \rangle$  obeying  $(\lambda, \bar{\lambda}, J_\kappa^{\text{bd}}, \kappa, \bar{\mathcal{P}})$ . Let  $\text{cd} : {}^\kappa\mu \rightarrow \mu$  be one-to-one, we may assume that  $(\forall i)\lambda_i > \kappa$  and  $\nu \in \prod_{j < \kappa} \lambda_j \wedge i < j < \kappa \Rightarrow \text{cd}(\nu \upharpoonright i) < \text{cd}(\nu \upharpoonright j)$ . Define  $f_\alpha^* : \kappa \rightarrow \mu$  by  $f_\alpha^*(i) = \text{cd}(f_\alpha \upharpoonright (i+1))$ , so  $f_\alpha^*$  is increasing.

Lastly, let  $\alpha_{\delta,i,j} = \alpha_{\delta,i} + f_\delta^*(j)$  and we should prove that  $\langle \alpha_{\delta,i,j} : \delta \in S, i < \sigma, j < \kappa \rangle$  is as required in Definition 0.6, so  $\eta_\delta = \langle \alpha_{\delta,i,j} : (i,j) \in \sigma \times \kappa \rangle$ . If we have used  $f_\alpha^1$  instead of  $f_\alpha^*$  we just have to omit clause (d) of 0.6.

Clauses (a),(c) of 0.6 holds by our choice of  $\eta_\delta$ . Clause (b) of 0.6 holds by the choice of  $S$  noting that  $S \in \check{I}_\sigma[\lambda]$  as  $S \subseteq E \cap S_\sigma^\lambda$  and the case assumption. Clause (d) of 0.6 holds by the choices of the  $\bar{\alpha}_\delta$ 's and of  $\text{cd}, f_\alpha^*$  recalling  $f_\alpha^1 \in {}^\kappa\mu$  and  $\alpha_{\delta,i}$  is divisible by  $\mu$ . Clause (e) holds by 1.11, that is (B)(a) there says  $\bar{f} = \bar{f}^2$  is  $(\theta^{+\kappa+1}, \theta, J_*)$ -free when  $\theta \in [\kappa, \mu)$ . Also clause (f) of 0.6 that is " $\bar{f}$  is  $(\kappa^+, J_*)$ -free" holds by direct inspection or see clause (B)(a)(\beta) of 1.11.

Lastly, clause (g)' follows by clause (g) and clause (g) holds by [Sh:775].  $\square_{0.4}$

**Definition 1.13.** Let  $J$  be an ideal on  $\kappa$ .

1) We say  $\mathcal{F} \subseteq {}^\kappa\text{Ord}$  is strongly semi- $\langle \theta_2, \theta_1, J \rangle$ -stable there are no  $f_\varepsilon \in \mathcal{F}$  for  $\varepsilon < \theta_2$  and  $u \subseteq \text{Ord}$  of cardinality  $< \theta_1$  such that for  $\varepsilon < \zeta < \theta_2$  the following set  $A_{\varepsilon,\zeta} = A_{\kappa,\zeta}(u, \langle f_\varepsilon : \varepsilon \in u \rangle)$  is  $\neq \emptyset \pmod J$

$$A_{\varepsilon,\zeta} := \{i < \kappa : \min(u \cup \{\infty\} \setminus f_\varepsilon(i)) \neq \min(u \cup \{\infty\} \setminus f_\zeta(i))\}.$$

2) For  $<_J$ -increasing  $\bar{f} = \langle f_\alpha : \alpha < \alpha_* \rangle, f_\alpha \in {}^\kappa\text{Ord}$  we say  $\bar{f}$  is strongly-semi- $\langle \theta_2, \theta_1, J \rangle$ -stable (sequence) when there are no  $v \subseteq \alpha_*$  of cardinality  $\theta_2$  and  $u \subseteq \text{Ord}$  of cardinality  $< \theta_1$  such that: if  $\alpha < \beta$  are from  $v$  then the following set is  $\neq \emptyset \pmod J$

$$\{i < \kappa : \min(u \cup \{\infty\} \setminus f_\alpha(i)) \not\leq \min(u \cup \{\infty\} \setminus f_\beta(i))\}.$$

3) In parts (1),(2) above, if  $\theta_1 = \theta_2$  we may write  $(\theta, J)$  instead of  $(\theta_1, \theta_2)$ .

4) In parts (1),(2) above writing  $(\theta_2, \theta_1, J)$  instead of  $\langle \theta_2, \theta_1, J \rangle$  means: strongly-semi- $(\theta, J)$ -stable for every  $\theta \in [\theta_1, \theta_2)$ .

**Claim 1.14.** Assume  $\bar{f} = \langle f_\alpha : \alpha < \lambda \rangle$  witness the pcf-case  $(\lambda, \bar{\lambda}, J, \kappa)$  and is strongly-semi- $(\theta_2, \theta_1, J)$ -stable, see 1.13(2),(4). Then  $S_{\text{gd}}[\bar{f}] \supseteq \{\delta < \lambda : \text{cf}(\delta) \in [\theta_1, \theta_2)\}$ .

*Proof.* Straightforward.  $\square_{1.14}$

Note also

**Observation 1.15.** *Let  $J$  be an ideal on  $\kappa$ .*

1) *If  $f_\alpha \in {}^\kappa\text{Ord}$  for  $\alpha < \alpha_*$  and the sequence  $\langle f_\alpha : \alpha < \alpha_* \rangle$  is  $(\theta, J)$ -free then the set  $\{f_\alpha : \alpha < \alpha_*\}$  is  $(\theta, J)$ -free and is with no repetitions.*

2) *Similarly for  $(\theta_2, \theta_1, J)$ -free.*

2A) *Similarly for  $\langle \theta_2, \theta_1, J \rangle$ -free.*

3) *If  $\theta'_2 \geq \theta_2 \geq \theta_1 \geq \theta'_1$  then*

(a)  *$\mathcal{F}$  is  $(\theta_2, J)$ -free implies  $\mathcal{F}$  is  $(\theta_1, J)$ -free*

(b) *similarly for  $\bar{f}$*

(c)  *$\mathcal{F}$  is  $\langle \theta_2, \theta_1, J \rangle$ -stable implies  $\mathcal{F}$  is  $\langle \theta'_2, \theta'_1, J \rangle$ -stable.*

4) *If  $\mathcal{F} \subseteq {}^\kappa\text{Ord}$  is  $(\theta^+, J)$ -free then it is  $(\theta, J)$ -stable.*

5) *If  $\mathcal{F} \subseteq {}^\kappa\text{Ord}$  is  $(\theta_2^+, \theta_1, J)$ -free then  $\mathcal{F}$  is  $\langle \theta_2, \theta_1, J \rangle$ -free.*

6) *If  $\mathcal{F} \subseteq {}^\kappa\text{Ord}$  is  $\langle \theta_2, \theta_1, J \rangle$ -free then it is  $(\theta_2^+, \theta_1, J)$ -stable.*

*Remark 1.16.* We also have obvious monotonicity in  $\mathcal{F}$  and  $\bar{f}$  and other obvious implications.

**Claim 1.17.** 1) *Assume  $\mathcal{F} \subseteq {}^\kappa\text{Ord}$  is semi- $(\theta, J)$ -stable or just  $J$  is  $\theta_*$ -complete and  $\varepsilon \leq \theta$ . Then  $\mathcal{F}$  is semi- $(\theta^{+\varepsilon+1}, J)$ -stable.*

2) *Similarly without semi.*

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